

HÖLDER ESTIMATES FOR HOMOTOPY OPERATORS ON STRICTLY PSEUDOCONVEX DOMAINS WITH C^2 BOUNDARY

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ABSTRACT. We derive a new homotopy formula for a strictly pseudoconvex domain of C^2 boundary in \mathbf{C}^n by using a method of Lieb and Range and obtain estimates in Lipschitz spaces for the homotopy operators. For $r > 1$ and $q > 0$, we obtain a $\Lambda_{r+1/2}$ solution u to $\bar{\partial}u = f$ for a $\bar{\partial}$ -closed $(0, q)$ form f of class Λ_r in the domain. We apply the estimates to obtain boundary regularities of \mathcal{D} -solutions for a domain in the Levi-flat Euclidean space.

1. INTRODUCTION

The main purpose of this paper is to show the boundary regularity for $\bar{\partial}$ solutions in a strictly pseudoconvex domain D in \mathbf{C}^n under the minimal smoothness condition of the boundary $\partial D \in C^2$. We will also derive a homotopy formula for the domain D ,

$$(1.1) \quad \varphi = \bar{\partial}H_q\varphi + H_{q+1}\bar{\partial}\varphi, \quad q \geq 1$$

that admits a derivative estimate. Here φ is a $(0, q)$ -form in \bar{D} and $\varphi, \bar{\partial}\varphi$ are in $C^1(\bar{D})$. We will prove the following $C^{r+1/2}$ estimate.

Theorem 1.1. *Let $r \in [1, \infty)$ and $1 \leq q \leq n$. Let D be a bounded strictly pseudoconvex domain of C^2 boundary in \mathbf{C}^n . If $r + 1/2$ is non integral, then*

$$(1.2) \quad |H_q\varphi|_{C^{r+1/2}(\bar{D})} \leq C_r(D)|\varphi|_{C^r(\bar{D})},$$

where $C_r(D) < \infty$ depends only on r and the domain D .

Note that we do not require that φ is $\bar{\partial}$ -closed. Until now it has been an open problem if there exists a homotopy formula for such a boundary regularity to hold. An interior estimate of gaining one derivative for $\varphi \in C^r$ with a non-integral r was obtained by Webster [53]. An exception case is $r = 0$ for $\varphi \in L^\infty(D)$. Grauert-Lieb [22] and Lieb [33] obtained the sup-norm estimate. Kerzman [29] obtained a C^β -estimate for all $\beta < 1/2$. Finally, Henkin-Romanov [27] achieved the $1/2$ -estimate for their homotopy operator T_q [25, 43], and the $1/2$ -estimate is optimal for $\varphi \in C^0$ by an example of Stein [26]. Note that Treves [50] has studied the boundary regularity for the Leray-Koppelman homotopy operator. Although the $C^{1/2}$ estimate of Henkin-Romanov does not need φ to be $\bar{\partial}$ -closed, to the author's best knowledge it remains open if the T_q that acts on $(0, q)$ forms has a boundary regularity beyond the $C^{1/2}$ -estimate when the forms are not $\bar{\partial}$ -closed, including the case $\partial D \in C^\infty$. There are, of course, important results under the conditions that φ is $\bar{\partial}$ -closed and r is a positive integer k : Siu [47] proved the $C^{k+1/2}$ estimate for T_1 and Alt [4] obtained analogous results for the two $\bar{\partial}$ -solution operators of Kerzman [29] and Grauert-Lieb [22] for $(0, 1)$ forms. For $(0, q)$ -forms with $q \geq 1$, Lieb and Range [34] constructed a new homotopy formula H_q and proved

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(1.2) for $\partial D \in C^{k+2}$, and in [35, 36] they also showed that Kohn's canonical solution u to $\bar{\partial}u = \varphi$ is in $C^{k+1/2}(\bar{D})$ when $\varphi \in C^k(\bar{D})$ and $\partial D \in C^\infty$. The above-mentioned results are for strictly pseudoconvex domains. There are positive results for $\bar{\partial}$ solutions when D is a convex domain of D'Angelo finite type m : Diederich-Fornæss-Wiegerlinck [12] obtained the $C^{1/m}$ estimate for ellipsoids, Diederich-Fisher-Fornæss [11] and Cumenge [9] obtained the $C^{1/m}$ estimate, and Alexandre [3] achieved the $C^{k+1/m}$ estimate for $\bar{\partial}$ solutions. We mention that Alexandre [3] obtained the $C^{k+1/m}$ estimate for homotopy operators for the $\bar{\partial}_b$ -complex on the boundary of convex finite-type domains. For the $\bar{\partial}_b$ operator in a domain in a strictly pseudoconvex hypersurface M in \mathbf{C}^n with $n \geq 4$, the interior C^k estimate was obtained by Webster [52] and Ma-Michel [37] proved a boundary regularity for their homotopy operator. Gong-Webster [20] obtained an interior $C^{k+1/2}$ estimate when the M is in C^{k+2} .

We will derive a new homotopy operator for a strictly pseudoconvex domain D , by perfecting the formulation of the Lieb-Range $\bar{\partial}$ solution operator. The homotopy operator has the form

$$(1.3) \quad H_q \varphi(z) = \int_{\mathbf{C}^n} \Omega_{0,q-1}^0(z, \zeta) \wedge E\varphi(\zeta) + \int_{\mathbf{C}^n \setminus D} \Omega_{0,q-1}^{01}(z, \zeta) \wedge [\bar{\partial}, E]\varphi(\zeta)$$

for $z \in D$ and $q > 0$. Here $E: C(\bar{D}) \rightarrow C_0(\mathbf{C}^n)$ is a linear extension operator constructed by Stein [49], and it satisfies two important properties

$$|Ef|_{\mathbf{C}^n; r} \leq C_r |f|_{\bar{D}; r}, \quad |Ef|_{\Lambda_r(\mathbf{C}^n)} \leq C_r |f|_{\Lambda_r(\bar{D})}.$$

And $C^r(\bar{D})$ with norm $|\cdot|_{D; r}$ is the Hölder space; the $\Lambda_r(\bar{D})$ with norm $|\cdot|_{\Lambda_r(\bar{D})}$ is the Lipschitz space (see Definition 3.9). We mention two novel features in H_q : the first is a *regularized* Henkin-Ramírez function which we introduce for a strictly pseudoconvex domain with C^2 boundary, and the second is the commutator $[\bar{\partial}, E]$, defined by $[\bar{\partial}, E]\varphi = \bar{\partial}E\varphi - E\bar{\partial}\varphi$. The commutator has an important property:

$$[\bar{\partial}, E]f = 0, \quad \text{in } \bar{D}.$$

Combining with $[\bar{\partial}, E]: C^r(\bar{D}) \rightarrow C^{r-1}(\mathbf{C}^n)$, the commutator serves as a *smooth* cut-off operator losing one derivative. We mention closely related previous work. Lieb-Range [34] first introduced the Seeley extension operator for their $\bar{\partial}$ solutions operator for strictly pseudoconvex domains and the extension has been a basic technique in other situations. Ma-Michel [37] used it for a suitable domain in a strictly pseudoconvex real hypersurface in \mathbf{C}^n for $n \geq 4$ and Alexandre [3] use it for finite type convex domains. If φ is $\bar{\partial}$ closed, we obtain $[\bar{\partial}, E]\varphi = \bar{\partial}E\varphi$ and the H_q is an analogue of the Lieb-Range $\bar{\partial}$ solution operator. Furthermore, the commutator for $\bar{\partial}_b$ complex has also appeared in the thesis of Schaal; see [37, p. 69].

To the author's knowledge, all existing results on the smoothness of the $\bar{\partial}$ solutions require a better smoothness for the domains. Our results suggest that regarding the boundary regularity of $\bar{\partial}$ solutions concerns, it is the boundary geometry that matters. This seems to be a new type of results in several complex variables. A detailed version of Theorem 1.1, Theorem 5.2, yields the following.

Corollary 1.2. *Let $r > 1$ and $0 < q < n$. Let D be a bounded strictly pseudoconvex domain of C^2 boundary in \mathbf{C}^n . Let $\varphi \in \Lambda_r(\bar{D})$ be a $\bar{\partial}$ -closed $(0, q)$ in D . Then there is a solution $u \in \Lambda_{r+1/2}(\bar{D})$ to $\bar{\partial}u = \varphi$ in D .*

The case $q = n$, which is not included above, is actually quite simple. The domain needs not to be pseudoconvex and its boundary only needs minimal smoothness; see Proposition 3.13.

We will also obtain a new Bochner-Martinelli-Leray-Koppelman formula: If f is a C^1 function in \overline{D} with $\overline{\partial}f \in C^1(\overline{D})$, then

$$f = H_0 f + H_1 \overline{\partial} f,$$

where D is strictly pseudoconvex with C^2 boundary and

$$H_0 f = \int_{\mathbf{C}^n \setminus D} \Omega_{0,0}^1 \wedge [\overline{\partial}, E] f.$$

Here $\Omega_{0,0}^1$ is a Cauchy-Fantappiè form of the above-mentioned regularized Henkin-Ramirez function. In connections with previous work, $H_0 f$ is a holomorphic projection analogous to $\tilde{H}_0 f = \int_{\partial D} \Omega_{0,0}^1 f$. We will show in Theorem 5.2 that the holomorphic projection H_0 maps $\Lambda_r(\partial D)$ continuously into itself, when $r > 1$. For \tilde{H}_0 , Elgueta [13] obtained a similar estimate with a minor loss of regularity and Ahern-Schneider [1] obtained a sharp estimate that actually holds for all $r > 0$. See also Phong-Stein [42] for the regularity of Bergman and Szegő projections for strictly pseudoconvex domains with C^∞ boundary.

As mentioned earlier, one of our main results is a homotopy formula in (1.1) and (1.3), which admits Hölder estimates in \overline{D} . Using H_q , we will study the elliptic differential

$$\mathcal{D} := \overline{\partial}_z + d_t$$

in $(z, t) \in \mathbf{C}^n \times \mathbf{R}^m$, introduced by Treves [50]. Let $D \times S$ be a product domain in $\mathbf{C}^n \times \mathbf{R}^m$. A k -form φ in $D \times S$ is said of *mixed type* $(0, k)$ if

$$\varphi(z, t) = [\varphi]_0(z, t) + \cdots + [\varphi]_k(z, t)$$

where $[\varphi]_i$ has type $(0, i)$ in z and degree $k - i$ in t . By a \mathcal{D} -closed form φ , we mean $\mathcal{D}\varphi = 0$ and we have the following.

Theorem 1.3. *Let $1 \leq r \leq \infty$. Let D be a bounded strictly pseudoconvex domain with C^2 boundary and let S be a bounded star-shaped domain in \mathbf{R}^m . Let φ be a \mathcal{D} -closed form of mixed type $(0, k)$ in $D \times S$ with $k \geq 1$. If $\varphi \in C^r(\overline{D} \times \overline{S})$, there is a solution $u \in C^r(\overline{D} \times \overline{S})$ to $\mathcal{D}u = \varphi$.*

Hanges and Jacobowitz [24] proved the interior C^∞ regularity of the \mathcal{D} -solutions on a smooth domain Ω in $\mathbf{C}^n \times \mathbf{R}^m$ under a Levi strictly convex condition.

We further mention some important results concerning $\overline{\partial}$ or $\overline{\partial}_b$ solutions. The C^∞ regularity results of $\overline{\partial}$ solutions were achieved by Kohn [31] for smoothly bounded strictly pseudoconvex domains, by Kohn [31] for $n = 2$ and Catlin [6] for pseudoconvex domains of finite D'Angelo type [10], and by Kohn for smoothly bounded pseudoconvex domains [32]. McNeal [38] obtained exact subelliptic estimates for finite type convex domains. The results of finite smoothness solutions have also been obtained. For $(0, 1)$ forms, Corollary 1.2 with $\partial D \in C^\infty$ was proved by Phong-Stein [42] for \mathbf{C}^2 and Greiner-Stein [23] for \mathbf{C}^n . The analogous results for $(0, 1)$ forms were obtained by Nagel-Rosay-Stein-Wainger [40] for $\overline{\partial}_b$ for finite type pseudoconvex domains in \mathbf{C}^2 , by Fefferman-Kohn [14] and Christ [8] for $\overline{\partial}$ (resp. $\overline{\partial}_b$) solutions for finite type pseudoconvex domains (resp. boundary) in \mathbf{C}^2 , and by Fefferman-Kohn-Machedon [15] for finite type domains in \mathbf{C}^n with diagonalizable Levi form for $\overline{\partial}$ and $\overline{\partial}_b$ solutions of $(0, 1)$ forms. Note that Shaw [46] obtained the exact $1/m$ -Hölder

estimate of $\bar{\partial}_b$ -solutions for $(0, 1)$ -forms for the boundary of ellipsoids of finite type m in \mathbf{C}^n . For $(0, q)$ forms, the Hölder estimates for $\bar{\partial}_b$ solutions were finally achieved by Koenig [30] for finite type CR manifolds with diagonalizable Levi form.

We now state two questions.

Question 1. *Let $0 < q < n$. Let D be a bounded strictly pseudoconvex domain with C^2 boundary in \mathbf{C}^n . Let φ be a $\bar{\partial}$ -closed $(0, q)$ form in D . If $\varphi \in \Lambda_r(\bar{D})$ and $0 < r \leq 1$, does there exist $u \in \Lambda_{r+1/2}(\bar{D})$ satisfying $\bar{\partial}u = \varphi$ in D ?*

The above question is open for the case $\partial D \in C^\infty$. The case $q = 1$ for $\partial D \in C^\infty$ already has a positive result in Phong-Stein [42] and Greiner-Stein [23], while Corollary 1.2 gives a positive answer for the case $r > 1$. The result in Kohn [32] and Corollary 1.2 give rise to the following question.

Question 2. *Let $0 < q < n$. Let D be a bounded pseudoconvex domain in \mathbf{C}^n with C^2 boundary. Let φ be a $\bar{\partial}$ -closed $(0, q)$ form in D . If $\varphi \in C^\infty(\bar{D})$, does there exist $u \in C^\infty(\bar{D})$ satisfying $\bar{\partial}u = \varphi$ in D ?*

The paper is organized as follows. In section 2 we derive the homotopy formula. In section 3 we recall the Whitney and Stein extension operators from [49] and use them to obtain regularized defining functions for domains with C^2 boundary and describe equivalent norms of $\Lambda_r(\bar{D})$. Section 4 contains the main estimation of this paper, assuming the existence of a regularized Henkin-Ramírez function. The latter is derived in section 5 for which we follow the standard construction of Henkin-Ramírez functions. The final section contains two homotopy formulae for the \mathcal{D} -complex and the proof of Theorem 1.3.

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2. THE HOMOTOPY FORMULA AND THE COMMUTATOR

In this section we derive the homotopy formula under the assumption that the domain admits a C^1 Leray map.

We first recall the Leray-Koppelman homotopy formula. Let D be a bounded domain with C^1 boundary. Let $g^1: D \times \partial D \rightarrow \mathbf{C}^n$ be a C^1 mapping satisfying

$$g^1(z, \zeta) \cdot (\zeta - z) \neq 0, \quad \forall \zeta \in \partial D, z \in D.$$

Let $g^0(z, \zeta) = \bar{\zeta} - \bar{z}$ and $w = \zeta - z$. Define

$$\begin{aligned} \omega^i &= \frac{1}{2\pi i} \frac{g^i \cdot dw}{g^i \cdot w}, \quad \Omega^i = \omega^i \wedge (\bar{\partial}\omega^i)^{n-1}, \\ \Omega^{01} &= \omega^0 \wedge \omega^1 \wedge \sum_{\alpha+\beta=n-2} (\bar{\partial}\omega^0)^\alpha \wedge (\bar{\partial}\omega^1)^\beta. \end{aligned}$$

Here both differentials d and $\bar{\partial}$ are in z, ζ variables. We have

$$\omega^i \wedge (\bar{\partial}\omega^i)^\alpha = \frac{g^i \cdot dw \wedge (\bar{\partial}(g^i \cdot dw))^\alpha}{(2\pi i g^\ell \cdot w)^{\alpha+1}}, \quad \alpha = 1, 2, \dots$$

With $\Omega_{0,-1}^{01} = 0$, we decompose $\Omega^i = \sum \Omega_{0,q}^i$ and $\Omega^{01} = \sum \Omega_{0,q}^{01}$, where $\Omega_{0,q}^i$ (resp. $\Omega_{0,q}^{01}$) is of type $(0, q)$ in z , and types $(n, n-1-q)$ (resp. $(n, n-2-q)$) in ζ . By the Koppelman lemma [7, p. 204], we have

$$(2.1) \quad \bar{\partial}_\zeta \Omega_{0,q}^0 + \bar{\partial}_z \Omega_{0,q-1}^0 = 0, \quad q \geq 1,$$

$$(2.2) \quad \bar{\partial}_\zeta \Omega_{0,q}^{01} + \bar{\partial}_z \Omega_{0,q-1}^{01} = \Omega_{0,q}^0 - \Omega_{0,q}^1, \quad q \geq 0.$$

We need to know how a sign changes, when the exterior differential interchanges with integration. Following notations in Chen-Shaw [7, p. 266], we define

$$\int_{y \in M} u(x, y) dy^J \wedge dx^I = \left\{ \int_{y \in M} u(x, y) dy^J \right\} dx^I$$

for a continuous function u in a manifold M . If d_x is the exterior differential in x -variables, we have

$$(2.3) \quad d_x \int_M \phi(x, y) = (-1)^{\dim M} \int_M d_x \phi(x, y).$$

The Leray-Koppelman homotopy formula [7, p. 273] for a $(0, q)$ -form φ is given by

$$(2.4) \quad \varphi(z) = \bar{\partial}_z T_q \varphi + T_{q+1} \bar{\partial}_z \varphi, \quad z \in D, \quad 1 \leq q \leq n,$$

$$(2.5) \quad \varphi(z) = \int_{\partial D} \Omega_{0,0}^1 \varphi + T_1 \bar{\partial} \varphi, \quad q = 0,$$

with

$$(2.6) \quad T_q \varphi = - \int_{\partial D} \Omega_{0,q-1}^{01} \wedge \varphi + \int_D \Omega_{0,q-1}^0 \wedge \varphi, \quad q \geq 1.$$

Proposition 2.1. *Let $D \subset \mathbf{C}^n$ be a domain with C^1 boundary and let \mathcal{U} be a bounded neighborhood of \bar{D} . Let $g^0(z, \zeta) = \bar{\zeta} - \bar{z}$. Let $g^1(z, \zeta) = W(z, \zeta)$ where $W \in C^1(D \times (\mathcal{U} \setminus D))$ is a Leray mapping, that is that W is holomorphic in $z \in \mathcal{U}$ and satisfies*

$$\Phi(z, \zeta) := W(z, \zeta) \cdot (\zeta - z) \neq 0, \quad z \in D, \quad \zeta \in \mathcal{U} \setminus D.$$

Let φ be a $(0, q)$ form in \bar{D} . Suppose that φ and $\bar{\partial} \varphi$ are in $C^1(\bar{D})$. Then

$$(2.7) \quad \varphi = \bar{\partial} H_q \varphi + H_{q+1} \bar{\partial} \varphi, \quad 1 \leq q \leq n,$$

$$(2.8) \quad \varphi = H_0 \varphi + H_1 \bar{\partial} \varphi, \quad q = 0,$$

with

$$(2.9) \quad H_q \varphi = \int_{\mathcal{U}} \Omega_{0,q-1}^0 \wedge E \varphi + \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^{01} \wedge [\bar{\partial}, E] \varphi, \quad q > 0,$$

$$(2.10) \quad H_0 \varphi = \int_{\partial D} \Omega_{0,0}^1 \varphi - \int_{\mathcal{U} \setminus D} \Omega_{0,0}^1 \wedge E \bar{\partial} \varphi = \int_{\mathcal{U} \setminus D} \Omega_{0,0}^1 \wedge [\bar{\partial}, E] \varphi.$$

Proof. In the formulae, the extension E constructed in [49] will be recalled in Lemma 3.11 below. The E is defined for functions. We thus define $E\varphi$ by applying E componentwise to its coefficients, which results in a form of the same type. We may assume that $E\varphi$ has a compact support in \mathcal{U} , by using a cut-off function.

Assume that $q \geq 1$. Let us modify the solution operator T_q given by (2.4)-(2.6), by applying the method of Lieb-Range [34] via the linear extension E . The Ω^{01} has total degree $2n - 2$. Applying Stokes' formula and (2.2)-(2.3), we get

$$\begin{aligned}
(2.11) \quad & - \int_{\zeta \in \partial D} \Omega_{0,q-1}^{01} \wedge \varphi = \int_{\zeta \in \mathcal{U} \setminus D} \Omega_{0,q-1}^{01} \wedge \bar{\partial}_\zeta E\varphi + \int_{\zeta \in \mathcal{U} \setminus D} \bar{\partial}_\zeta \Omega_{0,q-1}^{01} \wedge E\varphi \\
& = \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^{01} \wedge \bar{\partial} E\varphi \\
& \quad - \int_{\mathcal{U} \setminus D} (\bar{\partial}_z \Omega_{0,q-2}^{01} \wedge E\varphi + \Omega_{0,q-1}^1 \wedge E\varphi - \Omega_{0,q-1}^0 \wedge E\varphi) \\
& = \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^{01} \wedge \bar{\partial} E\varphi - \bar{\partial}_z \int_{\mathcal{U} \setminus D} \Omega_{0,q-2}^{01} \wedge E\varphi \\
& \quad + \int_{\mathcal{U} \setminus D} (-\Omega_{0,q-1}^1 \wedge E\varphi + \Omega_{0,q-1}^0 \wedge E\varphi).
\end{aligned}$$

Let us apply $\bar{\partial}$ to the last 4 terms. The second of the four terms becomes zero. The third also becomes zero since it is holomorphic for $q = 1$ and it is zero for $q > 1$. Thus we obtain for $z \in D$

$$\begin{aligned}
(2.12) \quad & - \bar{\partial} \int_{\zeta \in \partial D} \Omega_{0,q-1}^{01}(z, \zeta) \wedge \varphi(\zeta) + \bar{\partial} \int_{\zeta \in D} \Omega_{0,q-1}^0(z, \zeta) \wedge \varphi(\zeta) \\
& = \bar{\partial} \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^{01}(z, \zeta) \wedge \bar{\partial} E\varphi(\zeta) + \bar{\partial} \int_{\mathcal{U}} \Omega_{0,q-1}^0(z, \zeta) \wedge E\varphi(\zeta).
\end{aligned}$$

So far we have used $\varphi \in C^1(\bar{D})$. Assume now that $\bar{\partial}\varphi \in C^1(\bar{D})$. Using the last 4 terms in (2.11) in which φ is replaced by $\bar{\partial}\varphi$, we obtain

$$\begin{aligned}
(2.13) \quad & - \int_{\partial D} \Omega_{0,q}^{01} \wedge \bar{\partial}\varphi + \int_D \Omega_{0,q}^0 \wedge \bar{\partial}\varphi = \int_{\mathcal{U} \setminus D} \Omega_{0,q}^{01} \wedge \bar{\partial} E\bar{\partial}\varphi \\
& \quad - \bar{\partial} \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^{01} \wedge E\bar{\partial}\varphi - \int_{\mathcal{U} \setminus D} \Omega_{0,q}^1 \wedge E\bar{\partial}\varphi \\
& \quad + \int_{\mathcal{U} \setminus D} \Omega_{0,q}^0 \wedge E\bar{\partial}\varphi + \int_D \Omega_{0,q}^0 \wedge \bar{\partial}\varphi.
\end{aligned}$$

On the right-hand side, the first term can be written via the commutator as $\bar{\partial} E \bar{\partial} \varphi = (\bar{\partial} E - E \bar{\partial}) \bar{\partial} \varphi$. Since $q \geq 1$, the third is zero. The second, when combined with the first term on the right-hand side of (2.12), gives us the desired commutator for φ . Adding (2.12)-(2.13) yields (2.7).

To derive (2.10), we get by (2.13)

$$\begin{aligned}
& - \int_{\partial D} \Omega_{0,0}^{01} \wedge \bar{\partial}\varphi + \int_D \Omega_{0,0}^0 \wedge \bar{\partial}\varphi = \int_{\mathcal{U} \setminus D} \Omega_{0,0}^{01} \wedge \bar{\partial} E \bar{\partial}\varphi \\
& \quad - \int_{\mathcal{U} \setminus D} \Omega_{0,1}^1 \wedge E \bar{\partial}\varphi + \int_{\mathcal{U}} \Omega_{0,1}^0 \wedge E \bar{\partial}\varphi = H_0 \varphi - \int_{\mathcal{U} \setminus D} \Omega_{0,1}^1 \wedge E \bar{\partial}\varphi.
\end{aligned}$$

Thus we have verified (2.10). \square

Throughout the paper, $|\cdot|_{D;r}$, or $|\cdot|_r$ for abbreviation, denotes the Hölder C^r norm for $r \in [0, \infty)$ on a domain D . We finish the section with the following interior estimate of Webster [53].

Proposition 2.2. *Let $r \in [0, \infty)$. Let \mathcal{U} be a bounded domain in \mathbf{C}^n with $\overline{D} \subset \mathcal{U}$. Let $L\psi = \int_{\mathcal{U}} \Omega_{0,q}^0 \wedge \psi$ with $0 \leq q \leq n$. Then*

$$(2.14) \quad |L\psi|_{\overline{D};r} \leq C_a^* |\psi|_{\mathcal{U};r},$$

$$(2.15) \quad |L\psi|_{\overline{D};r+1} \leq C_r^* |\psi|_{\mathcal{U};r}, \quad r \notin \mathbf{N},$$

where $C_r^* \leq C_r(\mathcal{U}) \text{dist}(D, \partial\mathcal{U})^{-c_0r-c_1}$ and $C_r(\mathcal{U})$ depends only on r and the diameter of \mathcal{U} .

3. REGULARIZED DEFINING FUNCTIONS AND PRELIMINARIES FOR LIPSCHITZ ESTIMATES.

In this section we define a *regularized* defining function for a domain D by Whitney's extension so that the derivatives of the extension have optimal growth rates near the boundary of D . The defining function will play an important role in our estimates. We recall an extension operator of Stein [49] and basic facts about Lipschitz spaces Λ_r and its equivalent norms. The equivalent norms are used for the $\Lambda_{r+1/2}$ estimate when $r + 1/2 = 2, 3, \dots$. We will also recall some basic results on the real interpolation theory. The interpolation will be used for $C^{r+1/2}$ estimates when $r = 2, 3, \dots$, which also improves the regularity result of Lieb-Range. While the results of this sections might be known to the reader, we formulate them for the purpose of this paper. We will also specify the dependence of the various constants on the domains, which are used to address the stability of estimates of the homotopy operators in Theorem 5.2. We will conclude the section with a regularity result for the $\bar{\partial}$ equation of top type.

Let us first introduce some notations. For $r \in \mathbf{R}$, $[r]$ denotes the largest integer k satisfying $k \leq r$. For two sets A, B in \mathbf{R}^n , $\text{dist}(A, B)$ denotes $\inf\{|a - b| : a \in A, b \in B\}$. For $\alpha \in \mathbf{N}^n$, $\partial_x^\alpha f$ denotes the partial derivative $\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f(x)$ and $\partial_x^k f$ also denotes the set of all partial derivatives of order k . Let D be a domain in \mathbf{R}^n . Let $C^r(\overline{D})$ denote the set of functions f in D such that $\partial_x^\alpha f$ extend to functions $f^{(\alpha)} \in C^{a-[\alpha]}(\overline{D})$ for $|\alpha| := \alpha_1 + \dots + \alpha_n \leq r$. For a continuous function f in \overline{D} , $|f|_{\overline{D};0}$ denotes $|f|_{L^\infty(D)}$, while $|f|_{\overline{D};\alpha}$ also denotes the Hölder norm for $f \in C^r(\overline{D})$ when $0 < r < 1$. For $f \in C^r(\overline{D})$, define

$$|f|_{\overline{D};r} := \max_{|\alpha| \leq r} |f^{(\alpha)}|_{\overline{D};r-[\alpha]}.$$

3.1. Regularized defining functions. Let F be a closed set in \mathbf{R}^n and let $r \in (0, \infty)$. We recall the following definition from Whitney [54] and Stein [49].

Definition 3.1. Let F be a non-empty closed subset of \mathbf{R}^n . A function f in F is said in $C_w^r(F)$ in terms of the functions $f^{(\alpha)}$ in F for $\alpha \in \mathbf{N}^n$ and $|\alpha| \leq r$ if $f^{(0)} = f$, there is a finite constant A so that $|f^{(\alpha)}(x)| \leq A$ for all $x \in F$, while R_α , defined by

$$(3.1) \quad P_\alpha(x, p) := \sum_{|\beta|+|\alpha| \leq r} \frac{f^{(\alpha+\beta)}(p)}{\beta!} (x-p)^\beta, \quad R_\alpha(x, p) := f^{(\alpha)}(x) - P_\alpha(x, p)$$

has the properties: (i) $|R_\alpha(x, p)| \leq A|x-p|^{r-|\alpha|}$ for $x, p \in F$ and $|\alpha| \leq r$; (ii) when $r \in \mathbf{N}$, for each $p \in F$ and $\epsilon > 0$ there is $\delta > 0$ so that for $x, x' \in F$ with $|x-p| + |x'-p| < \delta$,

$$(3.2) \quad |R_\alpha(x', x)| \leq \epsilon |x' - x|^{[r]-|\alpha|}.$$

Let $|f|_{F;r}^w$ denote the infimum of the constants A for all possible choices of $f^{(\alpha)}$ for $0 < |\alpha| \leq r$.

Proposition 3.2. *Fix $r \in [0, \infty)$. Let F be a closed subset of \mathbf{R}^n . There is an extension operator $E_r: C_w^r(F) \rightarrow C^r(\mathbf{R}^n)$ so that $|E_r f|_{\mathbf{R}^n;r} \leq C_r |f|_{F;r}^w$. Moreover, for $x \in F^c := \mathbf{R}^n \setminus F$ with $d(x) := \text{dist}(x, F) < 1$,*

$$(3.3) \quad |\partial_x^\alpha E_r f - P_\alpha(x, x_*)| \leq C_r \sup_{|\beta| \leq r, x' \in \partial F, |x' - x_*| < 4d(x)} |R_\beta(x^*, x')| d(x)^{|\beta| - |\alpha|};$$

$$(3.4) \quad |\partial_x^k E_r f(x)| \leq C_r |f|_{D;r}^w (1 + d(x)^{r-k}), \quad x \in F^c, \quad k = 0, 1, 2, \dots,$$

where $d(x) = |x - x_*|$ with $x_* \in F$, and $P_\beta = 0$ for $|\beta| > r$.

Proof. The inequality (3.4) is proved in Stein [49, p. 178] and Glaeser [17] when $|\alpha| \leq r + 1$ and stated in Glaeser [17, p. 31] for all α . We present here a proof for the reader's convenience. Recall from [49, p. 169-170] the following properties: (i) There are $\varphi_k \in C_0^\infty(F^c)$ so that $\sum \varphi_k = 1$ in F^c , $0 \leq \varphi_k \leq 1$, and

$$(3.5) \quad |\partial_x^\alpha \varphi_k| \leq C_\alpha \text{dist}(x, F)^{-|\alpha|}, \quad \text{supp } \varphi_k \subset Q_k,$$

where Q_k are cubes with $\frac{1}{2} \text{diam } Q_k \leq \text{dist}(Q_k, F) \leq 5 \text{diam } Q_k$, and $F^c = \cup_k Q_k$. (ii) Each point in F^c is contained in at most N_0 of cubes Q_k . Here N_0, A_α are independent of F .

For each Q_k , fix $p_k \in F$ such that $\text{dist}(F, Q_k) = \text{dist}(p_k, Q_k)$. We choose $\{f^{(\alpha)}: |\alpha| \leq r\}$ so that the constant A in Definition 3.1 satisfies $A \leq 2|f|_r$. Let $P(x, p) = \sum_{|\alpha| \leq r} \frac{1}{\alpha!} f^{(\alpha)}(p)(x - p)^\alpha$. Define $E_r f = f$ in F and

$$E_r f(x) = \sum_i' P(x, p_i) \varphi_i(x), \quad x \in F^c.$$

Here the sum is over the i for which $\text{dist}(Q_i, F) < 1$. When $d(x) < 1$ and $x \in Q_i$, we have $\text{dist}(Q_i, F) < 1$. Thus we drop the prime, by assuming $d(x) < 1$. Then

$$(3.6) \quad P_\beta(x, p) - P_\beta(x, q) = \sum_{|\gamma| \leq r - |\beta|} R_{\beta+\gamma}(p, q) \frac{(x - p)^\gamma}{\gamma!}, \quad p, q \in F,$$

$$(3.7) \quad \partial_x^\beta P(x, p) = P_\beta(x, p), \quad p \in F.$$

For $x \in F^c$, we fix $x_* \in F$ such that $|x - x_*| = \text{dist}(x, F)$. Suppose that $|\alpha| > 0$. Then

$$(3.8) \quad \begin{aligned} \partial_x^\alpha E_r f &= \sum_{\beta, \alpha - \beta \in \mathbf{N}^n} \sum_k \binom{\alpha}{\beta} \partial_x^\beta P(x, p_k) \partial_x^{\alpha - \beta} \varphi_k \\ &= P_\alpha(x, x_*) + \sum_{\beta, \alpha - \beta \in \mathbf{N}^n} \sum_k \binom{\alpha}{\beta} (P_\beta(x, p_k) - P_\beta(x, x_*)) \partial_x^{\alpha - \beta} \varphi_k. \end{aligned}$$

We only need to consider the terms with $\partial_x^{\alpha - \beta} \varphi_k \neq 0$. Thus $x \in Q_k$ and there are at most N_0 of such φ_k 's. When $\partial_x^{\alpha - \beta} \varphi_k \neq 0$, we have $|x - p_k| \leq \text{diam}(Q_k) + \text{dist}(Q_k, F) \leq 3 \text{dist}(Q_k, F) \leq 3d(x)$. Then by (3.6)

$$(3.9) \quad |P_\beta(x, x_*) - P_\beta(x, p_k)| |\partial_x^{\alpha - \beta} \varphi_k| \leq C \sum_{|\gamma| \leq r - |\beta|} |R_{\beta+\gamma}(x_*, p_k)| d(x)^{|\beta| + |\gamma| - |\alpha|}.$$

Combining it with (3.8) yields

$$|\partial_x^\alpha E_r f - P_\alpha(x, x_*)| \leq C \sum_{0 \leq |\beta| \leq r} |R_\beta(x_*, p_k)| d(x)^{|\beta| - |\alpha|}.$$

We also have $|x_* - p_k| \leq |x_* - x| + |x - p_k| \leq 4d(x)$. Hence $|\partial_x^\alpha E f - P_\alpha(x, x_*)| \leq C'|f|_r d(x)^{r-|\alpha|}$. Also the continuity of $\partial^\alpha E_a f$ comes from

$$|\partial_x^\alpha E f - P_\alpha(x, x_*)| \leq C' \epsilon_{x, x_*} d(x)^{[r]-|\alpha|}$$

with $\epsilon_{x, x_*} \rightarrow 0$ as x tends to $x_0 \in \partial D$. We have proved (3.3), while (3.4) follows directly from (3.3). When $r \in \mathbf{N}$, (3.4) implies that $|E_r f|_r \leq C|f|_{\overline{D}; r}^w$. When $r > [r]$, (3.4) for $k = [r] + 1$ also implies that $|\partial^{[r]} E_r f|_{\mathbf{R}^n; r-[r]} \leq C|f|_{\overline{D}; r}$; see the proof in [49, Thm. 3, p. 173]. \square

Remark 3.3. When $F = \overline{D}$ for a domain D in \mathbf{R}^n and $f \in C_w^a(F)$, D is dense in F and $f^{(\alpha)}$ are uniquely determined by the values of f in D ; in fact $f^{(\alpha)} = \partial^\alpha f$ in D . In this case, the above E_r becomes linear for a fixed sequence p_k appeared in the above proof.

We first identify $C^r(\overline{D})$ with $C_w^r(\overline{D})$ under a mild condition on the domain D .

Lemma 3.4. *Let $r \geq 1$ and $L \geq 1$. Let D be a domain in \mathbf{R}^n . Assume that any two points p, q in \overline{D} can be connected by γ , a union of finitely many line segments in \overline{D} , so that γ has length at most $L|p - q|$ and $\gamma \cap \partial D$ is a finite set. Then $C^r(\overline{D}) = C_w^r(\overline{D})$ and $|f|_{\overline{D}; r} \leq |f|_{\overline{D}; r}^w \leq C_r L^r |f|_{\overline{D}; r}$.*

Proof. When $|\alpha| = [r]$, we have $R_\alpha(x, y) = \partial_x^\alpha f - \partial_y^\alpha f$. By the continuity of $\partial_x^\alpha f$ and the definition of Hölder ratio, we get $|R_\alpha(x, y)| \leq |f|_{D; r} |x - y|^{r-[r]}$ and (3.2) by the continuity of $\partial^\alpha f$.

Assume that $|\alpha| < [r]$. Let $\gamma: [0, 1] \rightarrow \overline{D}$ be a piecewise linear curve with $\gamma(0) = p, \gamma(1) = q$. Suppose that $\gamma(t) \in D$ and $|\gamma'(t)| \leq L|p - q|$ for $t \in (t_k, t_{k+1})$ with $t_0 = 0, \dots, t_N = 1$. Choose an increasing C^∞ function \hat{s} such that $\hat{s}(0) = 0, \hat{s}(1) = 1$, and all derivatives of \hat{s} vanish at 0, 1. Let $s(t) = t_k + (t_{k+1} - t_k)\hat{s}((t - t_k)/(t_{k+1} - t_k))$ for $t \in [t_k, t_{k+1}]$. Then $s(t_j) = t_j, s^{(\ell)}(t_j) = 0$ for all $\ell > 0$, and $0 \leq s'(t) \leq C$, where C is independent of t_i, N . Then $t \rightarrow \gamma(s(t))$ is a C^∞ curve connecting p, q . Let $\gamma(t)$ still denote $\gamma(s(t))$. We have $|\gamma'(t)| \leq CL|p - q|$.

Let $g(t) = R_\alpha(\gamma(t), p)$. Then g is $C^1([0, 1])$. We have

$$g(1) = \sum_i \int_0^1 \partial_{x_i}|_{x=\gamma(s_1)} R_\alpha(x, p) \gamma'_i(s_1) ds_1.$$

We also have $\partial_x^\beta R_\alpha(x, p)$ vanishes at $x = p$ if $|\beta| + |\alpha| \leq r$. Therefore,

$$g(1) = \sum \int_0^1 \cdots \int_0^{s_{i-1}} \partial_{x_{k_i}} \cdots \partial_{x_{k_1}} R_\alpha(\gamma(s_i), p) \gamma'_{k_1}(s_1) \cdots \gamma'_{k_i}(s_i) ds_i \cdots ds_1,$$

for summing over k_1, \dots, k_i with $k_1 + \cdots + k_i = i$ and $i = [r] - |\alpha|$. We obtain

$$|R_\alpha(q, p)| \leq C_i (CL|p - q|)^{[r]-|\alpha|} \max_{t \in [0, 1], |\beta| = [r]-|\alpha|} |\partial_x^\beta R_\alpha(\gamma(t), p)|.$$

Note that $\partial_x^\beta R_\alpha(x, p) = R_{\beta+\alpha}(x, p)$. The lemma is verified. \square

The proof also yields the following inequality.

Proposition 3.5. *Let D be as in Lemma 3.4. Let $P(x, p)$ be the Taylor polynomial of f of degree k about $p \in \overline{D}$. Then for $x, p \in \overline{D}$*

$$|f(x) - P(x, p)| \leq C_k (L|x - p|)^k \sup_{x', |\alpha|=k} |\partial^\alpha f(x') - \partial^\alpha f(p)|,$$

where $x' \in \overline{D}$ and $|x' - p| \leq L|x - p|$.

Definition 3.6 ([49], p. 189). Let D be a domain in \mathbf{R}^n . We say that ∂D is *minimally smooth* if the following conditions hold: There are positive numbers ϵ, N, M , and a sequence of open subsets U_1, U_2, \dots of \mathbf{R}^n so that the following hold:

- (i) If $x \in \partial D$, then $B(x, \epsilon) \subset U_i$; $B(x, \epsilon)$ is the ball of center x and radius ϵ .
- (ii) No point of \mathbf{R}^n is contained in more than N of the U_i 's.
- (iii) For each i there exists a domain D_i in \mathbf{R}^n , defined by $x_{n_i} > \varphi_i(x'_{n_i})$ for $x = (x_1, \dots, x_n)$, $x'_{n_i} = (x_1, \dots, \hat{x}_{n_i}, \dots, x_n)$ so that $U_i \cap D = U_i \cap D_i$ and

$$|\varphi_i(u) - \varphi_i(v)| \leq M|u - v|, \quad u, v \in \mathbf{R}^{n-1}.$$

We will denote by $C_r(D)$ a finite number depending on the above M, N, ϵ , and r .

A bounded domain has a Lipschitz boundary if and only if its boundary is minimally smooth.

Lemma 3.7. Let D be a bounded domain in \mathbf{R}^n with C^2 boundary. Let $\rho_0 \in C^2(\overline{D})$ with $\partial \rho_0 \neq 0$ in ∂D and $\rho_0 \leq 0$ in \overline{D} . There exists a real function $\rho \in C^2(\mathbf{R}^n)$, $\rho = \rho_0$ in \overline{D} , and for $x \in \mathbf{R}^n \setminus \overline{D}$ and $d(x) = \text{dist}(x, D) < 1$,

$$(3.10) \quad |\partial_x^i \rho| \leq C_i L^2 |\rho_0|_{\overline{D}, 2} (1 + d(x)^{2-i}), \quad i = 0, 1, 2, \dots,$$

$$(3.11) \quad |\partial \rho(x) - \partial \rho_0(x_*)| \leq CL^2 |x - x_*| \max_{y \in \overline{D}, |y - x_*| \leq 4L|x - x_*|} |\partial_y^2 \rho_0|,$$

$$(3.12) \quad |\partial^2 \rho(x) - \partial^2 \rho_0(x_*)| \leq CL^2 \omega_2(|x - x_*|),$$

where $x_* \in \partial D$, $|x - x_*| = \text{dist}(x, D)$, and

$$\omega_2(\delta) = \sup_{x' \in \overline{D}, x \in \partial D, |x' - x| \leq 4L\delta} |\partial^2 \rho_0(x') - \partial^2 \rho_0(x)|.$$

If $0 < d(x) < \min_{y \in \partial D} \{1, |\partial_y \rho_0| / (CL^2 |\partial^2 \rho_0|_0)\}$, then

$$(3.13) \quad |\partial_x \rho| \geq \frac{1}{2} \min_{y \in \partial D} |\partial_y \rho_0|, \quad \rho(x) \geq \frac{1}{2} \min_{y \in \partial D} |\partial_y \rho_0| d(x).$$

Proof. Applying (3.3) and Proposition 3.2 to $\rho = E_2 \rho_0$ and $F = \overline{D}$, we see that

$$|\rho(x)| \leq C |(\partial \rho(x_*), \partial^2 \rho(x_*))| |x - x_*| + C \sup_{x', |\alpha| \leq 2} |R_\alpha(x', x_*)| d(x)^{|\alpha|},$$

$$|\partial \rho(x) - \partial \rho(x_*)| \leq C |\partial^2 \rho(x_*)| |x - x_*| + C \sup_{x', |\alpha| \leq 2} |R_\alpha(x', x_*)| d(x)^{|\alpha|-1},$$

$$|\partial^2 \rho(x) - \partial^2 \rho(x_*)| \leq C \sup_{x', |\alpha| \leq 2} |R_\alpha(x', x_*)| d(x)^{|\alpha|-2},$$

where $x' \in \partial D$ and $|x' - x_*| \leq 4|x - x_*|$. By Proposition 3.5, we have for $|\alpha| \leq 2$

$$|R_\alpha(x', x_*)| \leq C(L|x' - x_*|)^{2-|\alpha|} \sup_{x'' \in \overline{D}, |x'' - x_*| \leq L|x' - x_*|} |\partial^2 \rho(x'') - \partial^2 \rho(x_*)|.$$

This gives us (3.11)-(3.12). We get (3.10) from (3.4) and $|E_2 \rho_0|_2 \leq CL^2 |r_0|_{\overline{D}, 2}$ by Lemma 3.4. The estimate (3.13) follows directly from (3.11). \square

We call the above r a *regularized C^2 defining function* of D . We will also need the following version of Stokes' theorem.

Lemma 3.8. *Let m be a positive integer, and let $b \in \mathbf{R}$. Let \mathcal{V} be a bounded domain in \mathbf{R}^n with C^1 boundary. Assume that B and S are functions in $C^1(\mathcal{V})$ and for $x \in \mathcal{V}$ and $i = 0, 1$,*

$$|\partial_x^i S| < C \operatorname{dist}(x, \partial\mathcal{V})^{m-i}, \quad |\partial_x^i B| \leq C(1 + \operatorname{dist}^{b-i}(x, \partial\mathcal{V})), \quad C = C(B, S) < \infty.$$

Assume further that $b + m > 0$. Then $\int_{\mathcal{V}} B(x) \partial_{x_j} S \, dx = - \int_{\mathcal{V}} S(x) \partial_{x_j} B \, dx$.

Proof. Let $d(x) = \operatorname{dist}(x, \partial\mathcal{V})$. We know that $B \partial S$ is Lebesgue integrable in \mathcal{V} , since $|B(x) \partial_{x_i} S| \leq C'(1 + \operatorname{dist}(x)^{m+b-1})$. Analogously, $S(x) \partial_{x_i} B$ is integrable in \mathcal{V} .

Let N_δ be the set of $x \in \mathcal{V}$ with $d(x) < \delta$. Take $\chi_\ell \in C_0^\infty(\mathcal{V} \setminus N_{1/\ell})$ so that $0 \leq \chi_\ell \leq 1$, $\chi_\ell = 1$ in $\mathcal{V} \setminus N_{2/\ell}$, and $|\partial_x \chi_\ell| \leq C d(x)^{-1}$. Then we have

$$\left| \int_{\mathcal{V}} \partial_{x_j} ((1 - \chi_\ell) B S) \, dx \right| \leq C \int_{N_{2/\ell}} (1 + d(x)^{m-1+b}) \, dx,$$

which tends to 0 as $\ell \rightarrow \infty$ as D is bounded and $\partial D \in C^1$. Also, $\int_{\mathcal{V}} ((1 - \chi_\ell) S(x) \partial_{x_j} B \, dx$, $\int_{\mathcal{V}} ((1 - \chi_\ell) B(x) \partial_{x_j} S \, dx$, and $\int_{\mathcal{V}} (S B)(x) \partial_{x_j} \chi_\ell \, dx$ tend to 0 as $\ell \rightarrow \infty$. By Stokes' theorem, we have

$$\int_{\mathcal{V}} (S B)(x) \partial_{x_j} \chi_\ell \, dx + \int_{\mathcal{V}} (\chi_\ell B)(x) \partial_{x_j} S \, dx + \int_{\mathcal{V}} (\chi_\ell S)(x) \partial_{x_j} B \, dx = 0.$$

Letting $\ell \rightarrow \infty$, we get the identity. \square

3.2. Equivalent norms. For $0 < r \leq 1$, the Lipschitz space $\Lambda_r(\mathbf{R}^n)$ is the set of functions $f \in L^\infty(\mathbf{R}^n)$ such that

$$|f|_{\Lambda_r(\mathbf{R}^n)} := |f|_{\mathbf{R}^n, r} := |f|_{L^\infty(\mathbf{R}^n)} + \sup_{y \in \mathbf{R}^n} \frac{|\Delta_y f|_{L^\infty(\mathbf{R}^n)}}{|y|^r}, \quad 0 < r < 1,$$

$$|f|_{\Lambda_1(\mathbf{R}^n)} := |f|_{L^\infty(\mathbf{R}^n)} + \sup_{y \in \mathbf{R}^n \setminus \{0\}} \frac{|\Delta_y^2 f|_{L^\infty(\mathbf{R}^n)}}{|y|}.$$

Here $\Delta_y f(x) := f(y + x) - f(x)$ and thus $\Delta_y^2 f(x) = f(x + 2y) + f(x) - 2f(x + y)$. When $r > 1$, we define $\Lambda_r(\mathbf{R}^n)$ to be the set of functions $f \in C^{[r]-1}(\mathbf{R}^n)$ so that

$$|f|_{\Lambda_r(\mathbf{R}^n)} := |f|_{L^\infty(\mathbf{R}^n)} + |\partial f|_{\Lambda_{r-1}(\mathbf{R}^n)} < \infty.$$

By [49, Prop. 8, p. 146], the $|f|_{\Lambda_r(\mathbf{R}^n)}$ is equivalent with the expression

$$|f|_{L^\infty(\mathbf{R}^n)} + \sup_{y \in \mathbf{R}^n} \frac{|\Delta_y^2 f|_{L^\infty(\mathbf{R}^n)}}{|y|^r},$$

for $0 < r < 2$. For a non-integral r , $|\cdot|_{\Lambda_r}$ is equivalent with the Hölder norm $|\cdot|_{\mathbf{R}^n, r}$ by [49, Prop. 8, p. 146].

Definition 3.9. Let F be a closed subset in \mathbf{R}^n . Let $r \in (0, \infty)$. We write $f \in \Lambda_r(F)$ if there exists $\tilde{f} \in \Lambda_r(\mathbf{R}^n)$ such that $\tilde{f}|_F = f$. Define $|f|_{\Lambda_r(F)}$ to be the infimum of $|\tilde{f}|_{\Lambda_r(\mathbf{R}^n)}$ for all such extensions \tilde{f} . Sometime $|f|_{\Lambda_r}$ denotes $|f|_{\Lambda_r(F)}$ for abbreviation.

We now discuss equivalent norms of the spaces Λ_r . The following lemma is in McNeal-Stein [39]. We need a quantified version.

Proposition 3.10. *Let $0 < r < \infty$. Then $f \in \Lambda_r(\mathbf{R}^n)$ if and only if there is a decomposition $f = \sum_{k \geq 0} g_k$ so that $g_k \in C^\infty(\mathbf{R}^n)$ and*

$$(3.14) \quad |\partial^i g_k|_{L^\infty(\mathbf{R}^n)} \leq A 2^{k(i-r)}, \quad i = 0, \dots, [r] + 1.$$

Furthermore, the smallest constant $A_r(f)$ of A is equivalent with $|f|_{\Lambda_r}$, i.e. $c_r A_r(f) \leq |f|_{\Lambda_r} \leq C_r A_r(f)$ for some positive numbers c_r, C_r independent of f .

Proof. The lemma is proved by Greiner-Stein [23, p. 142] for $0 < r < 1$. For $0 < r \leq 1$, the existence of decomposition is proved in [23, p. 145]. The decomposition is also valid for $r > 1$ since $g_k(x) = \int \varphi_k(t) f(x-t) dt$, each $\varphi_k \in C^\infty(\mathbf{R}^n)$ is rapidly decreasing, and hence $\partial_{x_j} g_k(x) = \int \varphi_k(t) \partial_{x_j} f(x-t) dt$.

Assume that $1 \leq r < 2$ and (3.14) holds. We have $|f|_{L^\infty} \leq \sum A 2^{-rk} \leq C_1 A$. We decompose

$$\Delta_y^2 f(x-y) = \sum_{k \leq N} \Delta_y^2 f_k(x-y) + \sum_{k > N} \Delta_y^2 f_k(x-y).$$

The two sums are bounded by

$$|y|^2 \sum_{k \leq N} |\partial^2 g_k|_{L^\infty} \leq C_r A |y|^2 2^{2N-rN}, \quad |y| \sum_{k > N} |\partial g_k|_{L^\infty} \leq A |y| 2^{N-Nr}.$$

When $|y| < 1$, choose a positive integer N so that $1 \leq |y| 2^N \leq 2$. Hence $|f|_{\Lambda_1} \leq C A$. Analogously, we can verify the proposition for $r \geq 2$. \square

We will use a linear extension operator from Stein [49] to prove the following.

Proposition 3.11. *Let D be a domain in \mathbf{R}^n where ∂D is minimally smooth. There is a continuous linear extension operator $E: C^0(\overline{D}) \rightarrow C^0(\mathbf{R}^n)$ so that $Ef = f$ on D*

$$(3.15) \quad |Ef|_{\Lambda_r(\mathbf{R}^n)} \leq C_r(D) |f|_{\Lambda_r(\overline{D})}, \quad \forall r \in (0, \infty),$$

$$(3.16) \quad |Ef|_{C^r(\mathbf{R}^n)} \leq C_r(D) |f|_{C^r(\overline{D})}, \quad \forall r \in [0, \infty).$$

Proof. We follow the proof in [49]. We first recall an extension for each D_i . To simplify the notation, we drop the i and assume $n_i = n$. Thus D is defined by $x_n > \varphi(x')$. Set $D^c = \mathbf{R}^n \setminus D$, $\overline{D}^c = \mathbf{R}^n \setminus \overline{D}$, and let $d(x)$ be the distance of x from D . By [49, Thm. 2, p. 171], there is a regularized distance function $\Delta \in C^\infty(\overline{D}^c)$ vanishing on ∂D so that

$$(3.17) \quad c_1 d(x) \leq \Delta(x) \leq c_2 d(x), \quad |\partial_x^\alpha \Delta(x)| \leq C_\alpha \delta^{1-|\alpha|}(x), \quad x \in \overline{D}^c.$$

Then choose a rapidly decreasing function $\psi \in C^\infty([1, \infty))$ so that

$$(3.18) \quad \int_1^\infty \psi(\lambda) d\lambda = 1, \quad \int_1^\infty \lambda^k \psi(\lambda) d\lambda = 0, \quad k = 1, 2, \dots$$

We have a linear extension operator

$$\mathfrak{E}f(x) = \int_1^\infty f(x', x_n + \lambda \delta^*(x)) \psi(\lambda) d\lambda, \quad x \in \overline{D}^c,$$

where $\delta^*(x) = c\Delta(x) \geq 2(\varphi(x') - x_n)$. We need the following estimate:

$$(3.19) \quad |\mathfrak{E}f|_{L_k^\infty(\mathbf{R}^n)} \leq C_k |f|_{L_k^\infty(D)},$$

where $L_k^\infty(D)$ is the set of functions f in D such that the distributional derivative $\partial^i f \in L^\infty(D)$ for all $i = 0, \dots, k$, and the constant C_k depends only on the upper bound of M and k, p ; see [49, Thm. 5', p. 181].

Assume now that $f \in \Lambda_r(\overline{D})$. We verify (3.15) by using an argument in Greiner-Stein [23, p.p. 146–147] in which D is a half-space. By the definition of $\Lambda_r(\overline{D})$, f has an extension $\tilde{f} \in \Lambda_r(\mathbf{R}^n)$ so that $|\tilde{f}|_{\Lambda_r} \leq 2|f|_{\Lambda_r}$. Take a decomposition $\tilde{f} = \sum f_j$ satisfying (3.14). By (3.19), we have $\mathfrak{E}f = \sum \mathfrak{E}(f_j|_D)$ and the decomposition satisfies (3.14), i.e. $|\mathfrak{E}(f_j|_D)|_{L_k^\infty} \leq C_k|f_j|_{L_k^\infty(D)} \leq 2C_k|f|_{\Lambda_r}2^{j(r-k)}$. This shows that $|\mathfrak{E}f|_{\Lambda_r(\mathbf{R}^n)} \leq C'|f|_{\Lambda_r(\overline{D})}$. We have verified (3.15) for $D = D_i$.

We now verify (3.16) for D_i . When r is an integer and $f \in C^r(\overline{D})$, we need only to verify, by (3.19), that $\partial^r \mathfrak{E}f$ is continuous in \mathbf{R}^n . And if $\alpha = r - [r] > 0$, we need to estimate the C^α norm of $g = \partial_x^{[r]} \mathfrak{E}f$. Let us consider the case $[r] = 0$ first. The continuity of $\mathfrak{E}f$ follows from the continuity of f , $|f|_{L^\infty} < \infty$, the rapidly decreasing property of ψ , and

$$\mathfrak{E}f(x) - f(x', x_n + \delta^*(x)) = \int_1^\infty R_0 f(x, \lambda) \psi(\lambda) d\lambda,$$

where $R_0(x, \lambda) := f(x', x_n + \lambda\delta^*(x)) - f(x', x_n + \delta^*(x))$. To estimate the Hölder ratio at two points u, v in \mathbf{R}^n , we may assume that u, v in D^c . Let $d = |u - v|$. Since δ^* vanishes on ∂D and $\partial_x \delta^*$ is bounded in \overline{D}^c , then connecting u, v by the line segment we show that $|\delta^*(u) - \delta^*(v)| \leq C|u - v|$. Computing the Hölder ratio of each term in R_0 , we obtain

$$|R_0(u, \lambda) - R_0(v, \lambda)| \leq C_1 \lambda^\alpha |f|_\alpha d^\alpha.$$

We have verified (3.16) for $0 \leq r < 1$.

For $k = [r] > 0$, a k -th derivative $\partial^k \mathfrak{E}f$ is a sum of $\mathfrak{E}\partial^k f$ and terms of the form

$$I(x) = \partial^{1+\ell_1} \delta^*(x) \cdots \partial^{1+\ell_i} \delta^*(x) \int_1^\infty \lambda^j \partial^j f(x', x_n + \lambda\delta^*(x)) \psi(\lambda) d\lambda,$$

with $j + \ell_1 + \cdots + \ell_i \leq k$ and $j > 0$ and $i \leq j$. By the result for $[r] = 0$, we know that $\mathfrak{E}\partial^k f$ is continuous and has the desired estimate in $|\cdot|_\alpha$ norm. With $j > 0$ and (3.18), we subtract the Taylor polynomial of $\partial^j f(x', x_n + \lambda\delta^*(x))$ in λ of degree $k - j$ about $\lambda = 1$ from $\partial^j f(x', x_n + \lambda\delta^*(x))$ and apply a Taylor remainder formula to express $I(x)$ as a linear combination of

$$\tilde{I}(x) = \eta(x) \int_1^\infty \lambda^j R_k f(x, \lambda) \psi(\lambda) d\lambda,$$

with $\eta(x) := \delta^*(x)^{k-j} \partial^{1+\ell_1} \delta^*(x) \cdots \partial^{1+\ell_i} \delta^*(x)$ and

$$R_k f(x, \lambda) := \int_1^\lambda (\lambda - \theta)^{k-j} \{ \partial^k f(x', x_n + \theta\delta^*(x)) - \partial^k f(x', x_n + \delta^*(x)) \} d\theta.$$

By (3.17), we have $|\eta(x)| \leq C_0$. It is now clear that \tilde{I} and hence I is continuous in \overline{D}^c and vanish in ∂D . Therefore $\mathfrak{E}f \in C^{[r]}(\mathbf{R}^n)$.

For the Hölder ratio, let $\alpha = r - [r]$. Take two points $x, x' \in \mathbf{R}^n \setminus D$. If $x' \in \partial D$, we have $R_k f(x', \lambda) = 0$ and

$$|\tilde{I}(x) - \tilde{I}(x')| \leq C'_0 |f|_{\overline{D}, r} \delta^*(x)^\alpha \leq C'_0 |f|_{\overline{D}, r} |x - x'|^\alpha.$$

Let $d = |x - x'|$ and let d_L be the distance from ∂D to the line segment L connecting x, x' . We consider two cases: (i) $d_L \leq d$; (ii) $d_L > d$. In the first case, we take $x'' \in \partial D$ with distance at most d from L . Then $|x - x''| \leq 2d$ and $|x' - x''| \leq 2d$. We get

$$\begin{aligned} |\tilde{I}(x) - \tilde{I}(x')| &\leq |\tilde{I}(x) - \tilde{I}(x'')| + |\tilde{I}(x'') - \tilde{I}(x')| \\ &\leq C|f|_\alpha (|x - x''|^\alpha + |x' - x''|^\alpha) \leq C'|f|_\alpha d^\alpha. \end{aligned}$$

In the second case, we have $|R_k f(x, \lambda) - R_k f(x', \lambda)| \leq C \lambda^{k+1} |f|_r |x - x'|^\alpha$. Thus $\tilde{\eta}(x) := \int_1^\infty \lambda^j R_k f(x, \lambda) \psi(\lambda) d\lambda$ satisfies

$$(3.20) \quad |\eta(x)(\tilde{\eta}(x) - \tilde{\eta}(x'))| \leq C |f|_r |x - x'|^\alpha.$$

By (3.17), we have $|\partial \eta(x)| \leq C_1 d(x)^{-1}$. Thus, $|\eta(x) - \eta(x')| \leq \sup_{\zeta \in L} |\partial_\zeta \eta| |x - x'| \leq C d_L^{-1} |x - x'|$ and $|\tilde{\eta}(x')| \leq C |f|_r d(x')^\alpha$, we obtain

$$(3.21) \quad |\tilde{\eta}(x')(\eta(x) - \eta(x'))| \leq C |f|_r d(x')^\alpha d_L^{-1} |x - x'| \leq C |f|_r d(x')^\alpha d_L^{-\alpha} |x - x'|^\alpha.$$

Furthermore, $d_L \geq d(x') - |x - x'| \geq d(x') - d_L$. We simplify (3.21) and combine it with (3.20) to conclude $|\tilde{I}(x) - \tilde{I}(x')| \leq C_r |f|_r |x - x'|^\alpha$ for the second case.

Therefore, we have verified (3.16) for each D_i . In the general case, we can verify that the linear extension operator \mathfrak{E} defined in [49, p. 191, formula (31)] satisfies (3.15)-(3.16). We leave the details to the reader. \square

3.3. Real interpolation. We recall some basic results of the real interpolation via the K -method of Peetre from Butzer-Berens [5, sect 3.2, p. 165]. Let X_0, X_1 be two Banach spaces embedded continuously in a linear Hausdorff space \mathcal{X} . Define

$$|f|_{X_0 \cap X_1} = \max\{|f|_{X_0}, |f|_{X_1}\}, \quad |f|_{X_0 + X_1} = \inf_{f=f_0+f_1} (|f_0|_{X_0} + |f_1|_{X_1}).$$

The (X_0, X_1) is called an interpolation pair of Banach spaces in \mathcal{X} . Define

$$K(t, f; X_0, X_1) = \inf_{f=f_0+f_1} (|f_0|_{X_0} + t|f_1|_{X_1}), \quad t > 0, \quad f \in X_0 + X_1.$$

Let $\theta \in (0, 1)$. By $f \in X_{\theta, \infty; K}$, we mean that

$$|f|_{\theta; X_0, X_1} := \sup_{t>0} \{t^{-\theta} K(t, f; X_0, X_1)\} < \infty.$$

Then $(X_0, X_1)_\theta := X_{\theta, \infty; K}$ with norm $|\cdot|_{\theta; X_0, X_1}$ is a Banach space, while $X_0 \cap X_1 \subset (X_0, X_1)_\theta \subset X_0 + X_1$ are continuous embeddings. Following Triebel [51, Sect. 2.7, p.p. 200-202], let $\mathcal{C}^r(\mathbf{R}^n)$ be the closure of the space of rapidly decreasing functions in \mathbf{R}^n in $\Lambda_r(\mathbf{R}^n)$. Then by [51, Thm. 1, p. 201] we have

$$(3.22) \quad (\mathcal{C}^{r_0}(\mathbf{R}^n), \mathcal{C}^{r_1}(\mathbf{R}^n))_\theta = \mathcal{C}^{(1-\theta)r_0 + \theta r_1}(\mathbf{R}^n), \quad 0 < \theta < 1, \quad 0 < r_0 < r_1 < \infty$$

in equivalent norms. Let (Y_0, Y_1) be an interpolation couple of Banach spaces continuously embedded in a linear Hausdorff space \mathcal{Y} . If $T: \mathcal{X} \rightarrow \mathcal{Y}$ is linear, and if

$$\|Tf_i\|_{Y_i} \leq M_i \|f_i\|_{X_i}, \quad i = 0, 1$$

then $\|Tf\|_{(Y_0, Y_1)_\theta} \leq M_0^{1-\theta} M_1^\theta \|f\|_{(X_0, X_1)_\theta}$; see [5, Thm. 3.2.23, p. 180] or [51, p. 26].

In summary, we can apply the following:

Proposition 3.12. *Let $\mathcal{C}^r = \mathcal{C}^r(\mathbf{R}^n)$. Let a_i, b_i are positive real numbers satisfying $a_0 < a_1$ and $b_0 < b_1$. Let $T: \mathcal{C}^{a_0} \rightarrow \mathcal{C}^{b_0}$ be a linear operator such that $|Tf|_{\mathcal{C}^{b_i}} \leq M_i |f|_{\mathcal{C}^{a_i}}$, for $i = 0, 1$. Then in equivalent norms, $|Tf|_{\mathcal{C}^{b_\theta}} \leq C_{r, b, \theta} M_0^{1-\theta} M_1^\theta |f|_{\mathcal{C}^{a_\theta}}$ for $0 < \theta < 1$, $a_\theta = (1-\theta)a_0 + \theta a_1$, and $b_\theta = (1-\theta)b_0 + \theta b_1$.*

3.4. $\bar{\partial}$ -solutions for top type. As an application of the extension and interpolation, we estimate a $\bar{\partial}$ -solution for forms of type $(0, n)$. Let $C_{(0,q)}^r(\bar{D})$ be the set of $(0, q)$ forms in D of which the coefficients are in $C^r(\bar{D})$. It seems that the statement has not appeared in the literature.

Proposition 3.13. *Let D be a bounded domain in \mathbf{C}^n .*

- (i) *Suppose that any two points p, q in D can be joined by a broken line segment γ in \bar{D} of length at most $L|p - q|$, while $\gamma \cap \partial D$ is a finite set. For each $r \in (0, \infty) \setminus \mathbf{N}$, there is a linear map $T: C_{0,n}^r(\bar{D}) \rightarrow C_{0,n-1}^{r+1}(\mathbf{C}^n)$ so that $\bar{\partial}T\varphi = \varphi$ in D and $|T\varphi|_{\mathbf{C}^n; r+1} \leq C_r(D)|\varphi|_{\bar{D}; r}$.*
- (ii) *Assume that ∂D is Lipschitz. There is a linear operator $S: C_{0,n}(\bar{D}) \rightarrow C_{0,n-1}(\mathbf{C}^n)$ so that $\bar{\partial}S\varphi = \varphi$ and $|S\varphi|_{\Lambda_{r+1}(\mathbf{C}^n)} \leq C_r(D)|\varphi|_{\Lambda_r(\bar{D})}$ with $C_r(D) < \infty$ for all $r \in (0, \infty)$.*

Proof. (i). We apply the Whitney extension E_r for \bar{D} via Lemma 3.4 and Proposition 3.2. Fix a ball of B so that $\text{dist}(\partial B, D) = 2 \text{diam}(D)$. By the Leray-Koppelman solution operator T_n for B and estimate in [53], we get the conclusion.

(ii). We first consider the case of a non integral r . We have $\varphi = f d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$. By (3.15), we may assume that D is relatively compact in a ball B_r while $\varphi \in \Lambda_r(\mathbf{C}^n)$. Take a sequence $\varphi_j \in C^\infty(\mathbf{C}^n)$ satisfying $\varphi_j \rightarrow \varphi$ in $L^\infty(D)$. Then we have $T_n\varphi_j \rightarrow T_n\varphi$ in $L^\infty(D)$. Since $\bar{\partial}T_n\varphi_j = \varphi_j$, we get $\bar{\partial}T_n\varphi = \varphi$ in the sense of distributions. By (2.15), we get $|T_n\varphi|_{\bar{D}; r+1} \leq C_r|\varphi|_{B_r; r}$. By (3.15) again, we conclude that

$$(3.23) \quad |ET_n\varphi|_{\mathbf{C}^n; r+1} \leq C_r|\varphi|_{B_r; r}, \quad r > 0.$$

When r is a positive integer, the estimate follows from interpolation by Proposition 3.12 as follows. Let $E: \Lambda_a(\bar{D}) \rightarrow \Lambda_a(\mathbf{C}^n)$. We consider a linear operator

$$\tilde{T}_n := \chi ET_n: \mathcal{C}_{(0,n)}(\mathbf{C}^n) \rightarrow \mathcal{C}_{(0,n-1)}(\mathbf{C}^n).$$

where $\chi \in C_0^\infty(\mathbf{C}^n)$ and $\chi = 1$ in B_r . By (3.23), we have $|\tilde{T}\varphi|_{\Lambda^{r+1}} \leq M_r|\varphi|_{\Lambda^r}$ for $r \in (0, \infty) \setminus \mathbf{N}$. By Proposition 3.12, we get the same estimate for all positive integer r . \square

Remark 3.14. The constant $C_r(D)$ in Proposition 3.13 (i) depends on L and the diameter of D and the $C_r(D)$ in (ii) depends on the constants ϵ, M, N in Definition 3.6, as well as the diameter of D by Proposition 2.2.

4. ESTIMATES FOR THE HOMOTOPY OPERATORS

In this section we first introduce a regularized Leray map to study strictly pseudoconvex domains with low regularity. The main estimates are derived under the assumption of the existence of a *regularized* Henkin-Ramírez for the homotopy operators H_q .

Definition 4.1. Let D be a bounded domain of class C^2 and define

$$D_\delta = \{z \in \mathbf{C}^n: \text{dist}(z, \bar{D}) < \delta\}, \quad D_{-\delta} = \{z \in D: \text{dist}(z, \partial D) > \delta\}, \quad \delta > 0.$$

We say that W is a *regularized* Leray mapping in $D_\delta \times (D_\delta \setminus D_{-\delta})$, if for some positive number δ the following hold

- (i) $W: D_\delta \times (D_\delta \setminus D_{-\delta}) \rightarrow \mathbf{C}^n$ is a C^1 mapping, and $W(z, \zeta)$ is holomorphic in $z \in D_\delta$.
- (ii) $W(z, \zeta) \cdot (\zeta - z) \neq 0$ for $z \in D$, $\zeta \in D_\delta \setminus D$ and $z \neq \zeta$.

(iii) For each $z \in D_\delta$, we have $W(z, \cdot) \in C^1(\overline{D} \setminus D_{-\delta})$ and

$$|\partial_\zeta^i W(z, \zeta)| \leq C_i |W(z, \cdot)|_{D,1} (1 + \text{dist}^{1-i}(\zeta, D)), \quad \zeta \in D_\delta \setminus \overline{D}, \quad 0 \leq i < \infty.$$

The first two properties are the standard requirements for the Leray maps. The third is new. Note that $W(z, \zeta)$ is C^∞ in $\mathbf{C}^n \setminus \overline{D}$ and its ζ -derivatives as ζ approaches to \overline{D} have precise growth rates.

The existence of a regularized C^2 defining function for a domain with C^2 boundary is proved in Lemma 3.7. The Whitney extension of a strictly convex function ρ in \overline{D} remains convex in a neighborhood of \overline{D} . Therefore, we have the following.

Example 4.2. Let D be defined by $\rho_0 < 0$ in \mathcal{U} with $\overline{D} \subset \mathcal{U}$. Suppose that ρ_0 is a C^2 strictly convex function in \mathcal{U} . Let ρ be a Whitney extension of $\rho_0|_{\overline{D}}$ as in Lemma 3.7. Then $W(z, \zeta) = (\rho_{\zeta_1}, \dots, \rho_{\zeta_n})$ is a regularized Leray mapping.

We now derive our main estimates. Recall the homotopy operator

$$H_q \varphi = \int_{\mathcal{U}} \Omega_{0,q-1}^0 \wedge E \varphi + \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^{01} \wedge [\overline{\partial}, E] \varphi.$$

The first term is estimated by Proposition 2.2, gaining one derivative in a Hölder space. We now estimate the second term for $z \in D$:

$$(4.1) \quad \int_{\mathcal{U} \setminus D} \Omega_{0,q}^{01}(z, \zeta) \wedge [\overline{\partial}, E] \varphi(\zeta).$$

From now on, we take $g^0(z, \zeta) = \overline{z} - \overline{\zeta}$ and $g^1(z, \zeta) = W(z, \zeta)$. We require that W is a regularized Leray map.

We will denote by ∂_z^k a derivative of order k in (z, \overline{z}) , and by $N_k(\zeta - z)$ a monomial in $\zeta - z, \overline{\zeta} - \overline{z}$ of degree k . Let $A(w)$ denote a polynomial w, \overline{w} , where N_k and A may differ when they recur. We can write (4.1) as a linear combination of

$$(4.2) \quad Kf(z) := \int_{\mathbf{C}^n} f(\zeta) \frac{A(\hat{\partial}_\zeta W(z, \zeta), z, \zeta) N_1(\zeta - z)}{\Phi^{n-\ell}(z, \zeta) |\zeta - z|^{2\ell}} dV(\zeta), \quad 1 \leq \ell < n,$$

$$(4.3) \quad \Phi(z, \zeta) = W(z, \zeta) \cdot (\zeta - z),$$

where f is a coefficient of the form $[\overline{\partial}, E] \varphi$. In particular f vanishes on \overline{D} . Here $\hat{\partial}_\zeta W$ denotes W and its first-order ζ derivatives.

To derive our main estimates, we start with the following lemma.

Lemma 4.3. Let $\beta \geq 0$, $\alpha \geq 0$, and $0 < \delta < 1/2$.

(i) If $\alpha < \beta + 1/2$, then $\int_0^1 \int_0^1 \frac{s^{\alpha+1} dt ds}{(\delta + s + t^2)^{3+\beta}} \leq C \delta^{\alpha - \frac{1}{2} - \beta}$.

(ii) $\int_{s=\delta}^{s=2\delta} \int_0^1 \frac{s^{\alpha+1} dt ds}{(s + t^2)^{1+\beta}} \leq C \delta^{\alpha - \beta + 3/2}$.

Proof. (i). We divide the domain in three regions

$$P: \delta + s \geq t, \quad Q: \delta + s \leq t^2, \quad R: t \geq \delta + s \geq t^2.$$

The integral in P is bounded above by

$$\int_{s=0}^1 \int_{t=0}^{\delta+s} \frac{s^{\alpha+1} dt ds}{(\delta + s)^{3+\beta}} \leq \int_0^1 \frac{s^{\alpha+1} ds}{(\delta + s)^{2+\beta}} \leq \int_0^1 (\delta + s)^{\alpha-1-\beta} ds,$$

which is less than $C\delta^{\alpha-\beta-1/8}$. In Q , it is bounded by

$$\int_{s=0}^1 \int_{t=\sqrt{\delta+s}}^1 \frac{s^{\alpha+1} dt ds}{t^{6+2\beta}} \leq \int_{s=0}^1 \frac{s^{\alpha+1}}{(\delta+s)^{\beta+5/2}} ds,$$

which is less than $C\delta^{\alpha-\beta-1/2}$. In R , it has a similar bound as

$$\int_{s=0}^1 \int_{t=0}^{\sqrt{\delta+s}} \frac{s^{\alpha+1} dt ds}{(\delta+s)^{3+\beta}} \leq \int_{s=0}^1 \frac{s^{\alpha+1} ds}{(\delta+s)^{\beta+5/2}}.$$

(ii). We divide the domain in 3 regions

$$P: s \geq t, \quad Q: s \leq t^2, \quad R: t \geq s \geq t^2, \quad t \leq \delta^{1/2}.$$

The integral in P is bounded above by $\int_{\delta}^{2\delta} s^{\alpha-\beta} \int_0^s dt ds \leq C\delta^{\alpha-\beta+2}$. In Q , it is bounded by $\int_{\delta}^{2\delta} s^{\alpha+1} \int_{\sqrt{s}}^1 t^{-2(\beta+1)} dt ds \leq C\delta^{\alpha-\beta+3/2}$. In R , it is bounded by $\int_{\delta}^{2\delta} s^{\alpha-\beta} \int_0^{\sqrt{s}} dt ds \leq C\delta^{\alpha-\beta+3/2}$. \square

Proposition 4.4. *Let $1 \leq r \leq \infty$. Let D be strictly pseudoconvex domain defined by $\rho < 0$ with $\rho \in C^2(\mathcal{U})$ with $\overline{D} \subset \mathcal{U}$. Suppose that $\partial\rho \neq 0$ in ∂D . Let $d(z)$ be the distance to z from ∂D . Let W be a regularized Leray mapping of D in $D_{\delta} \times (D_{\delta} \setminus D_{-\delta})$ and let Φ, Kf be defined by (4.2)-(4.3). Assume that there is a finite open covering $\{\omega_1, \dots, \omega_m\}$ of ∂D and C^1 coordinate maps $\Psi_i: \zeta \rightarrow (s, t) = (s_1, s_2, t_3, \dots, t_{2n})$ defined in ω_i such that $s_1 = \rho(\zeta)$ and for $z \in \omega_i \cap D, \zeta \in \omega_i \setminus D$*

$$(4.4) \quad |\Phi(z, \zeta)| \geq c_*(d(z) + s_1(\zeta) + |s_2(\zeta)| + |t(\zeta)|^2),$$

$$(4.5) \quad |\Phi(z, \zeta)| \geq c_*|\zeta - z|^2, \quad |\zeta - z| \geq c_*|t(\zeta)|,$$

where $c_* > 0$ is a constant. Suppose that $f \in C_0^{r-1}(\mathcal{U})$ vanishes in D . Then the following hold:

(i) If $r \geq 1$, then for $z \in D$ and $d(z) = \text{dist}(z, \partial D)$

$$|\partial_z^{[r]+1} Kf| \leq C_r d(z)^{\alpha-1/2} \|f\|_{\mathcal{U}; [r]-1/2}, \quad 0 < \alpha < 1/2,$$

$$|\partial_z^{[r]+2} Kf| \leq C_r d(z)^{\alpha-3/2} \|f\|_{\mathcal{U}; [r]-1/2}, \quad 1/2 \leq \alpha < 1.$$

(ii) $\|Kf\|_{\overline{D}, 3/2} \leq C_1 \|f\|_{\mathcal{U}, 0}$. For $r > 1$, $\|Kf\|_{\Lambda_{r+1/2}(\overline{D})} \leq C_r \|f\|_{\Lambda_{r-1}(\mathcal{U})}$.

Here C_r depends on $r, c_*, \sup_{z \in D_{\delta}} |W(z, \cdot)|_{D_{\delta} \setminus D_{-\delta}; 1}$, and C^1 norms of Ψ_i and the sup norms of $(\det \partial_{\zeta} \Psi_i)^{-1}$.

Proof. By the assumption, we have $\partial D \in C^1$. We have $\Phi(z, \zeta) \neq 0$, for $z \in D$ and $\zeta \in \mathcal{U} \setminus D$. The latter contains the support of f . We first consider the case that $f \in C^{r-1}$ and we distribute the first $[r] - 1$ derivatives in ζ directly on the integrand when $[r] > 1$. We then apply the integration by parts to derive a new formula.

We now explain how $\partial D \in C^2$ suffices the estimation.

(i) We write $\partial_z^{[r]-1} \{Kf(z)\}$ as a linear combination of $K_1 f(z)$ with

$$(4.6) \quad K_1 f(z) := \int_{\mathbf{C}^n \setminus \overline{D}} f_1(z, \zeta) \frac{N_{1-\mu_0+\mu_2}(\zeta - z)}{\Phi^{n-j+\mu_1}(z, \zeta) |\zeta - z|^{2j+2\mu_2}} dV(\zeta),$$

$$(4.7) \quad f_1(z, \zeta) = f(\zeta) A_1(W_1(z, \zeta), z, \zeta), \quad W_1(z, \zeta) = (\hat{\partial}_{\zeta} W(z, \zeta), \partial_z^{k_0} \hat{\partial}_{\zeta} W(z, \zeta)),$$

$$(4.8) \quad \mu_0 + \mu_1 + \mu_2 \leq [r] - 1, \quad 1 - \mu_0 + \mu_2 \geq 0,$$

where A_1 is a polynomial. Since $\hat{\partial}_\zeta W(z, \zeta)$ is holomorphic in z , its z -derivatives in a suitable neighborhood of $D_{\delta/2}$ can be estimated by the sup norm of $\hat{\partial}_\zeta W$ in D_δ by using the Cauchy formula. In $\mathbf{C}^n \setminus \overline{D}$, the integrand in $K_1 f$ is smooth in z . As $\zeta \in \mathbf{C}^n \setminus \overline{D}$ approaches ∂D , the rate of growth of a ζ -derivative of W_1 is bounded by a precise negative power of $d(\zeta)$. The latter can be dominated by the order of vanishing of $f(\zeta)$ along ∂D . Let us record the estimate

$$(4.9) \quad |\partial_\zeta^i \partial_z^j W_1(\zeta, z)| \leq C_{i+j}(|W|_1) d(\zeta)^{-i},$$

$$(4.10) \quad C_k(|W|_1) := C_k(\sup_{z \in D_\delta} |W(z, \cdot)|_{D_\delta \setminus D; 1}),$$

for $\zeta \in D_\delta \setminus \overline{D}$, $z \in D_{\delta/2}$ and $i, j = 0, 1, \dots$. Our argument relies essentially on that $f(\zeta)$ is independent of z .

We now provide the details of the proof. We will use integration by parts as in Elgueta [13], Ahern-Schneider [1], and Lieb-Range [34] to reduce the exponent of Φ to the original $n - j$. In our case, the integration by parts is carried out by applying Lemma 3.8 since $W_1(z, \zeta)$, which is C^∞ in $\zeta \in D_\delta \setminus \overline{D}$, is merely continuous in $\zeta \in D^c$. To this end we write (4.6) as

$$K_1 f(z) := \int_{\mathbf{C}^n \setminus \overline{D}} \frac{h_1(z, \zeta)}{\Phi^{n-j+\mu_1}(z, \zeta)} dV(\zeta),$$

with

$$h_1(z, \zeta) = f_1(z, \zeta) \frac{N_{1-\mu_0+\mu_2}(\zeta - z)}{|\zeta - z|^{2j+2\mu_2}}.$$

Using a partition of unity in ζ space and replacing f by χf for a C^∞ cut-off function, we may assume that

$$\text{supp } f \subset B_0 \setminus D, \quad u(z, \zeta) := \partial_{\zeta_{i^*}} \Phi(z, \zeta) \neq 0$$

for some i^* . Recall that $\partial D \in C^1$. We have for $\zeta \in B_0 \setminus D$

$$\begin{aligned} |\partial_\zeta^i W(z, \zeta)| &\leq C_i(|W|_1)(1 + d(\zeta)^{1-i}), \quad i = 0, 1, \dots, \\ |\partial_\zeta^i W_1(z, \zeta)| + |\partial_\zeta^i \frac{1}{u(z, \zeta)}| &\leq C_i(|W|_1) d(\zeta)^{-i}, \quad i = 0, 1, \dots \end{aligned}$$

Up to a constant multiple, we rewrite $K_1 f$ as

$$K_1 f = \int_{\mathbf{C}^n \setminus \overline{D}} u(z, \zeta)^{-1} h_1(z, \zeta) \partial_{\zeta_{i^*}} \Phi^{n-j+\mu_1}(z, \zeta) dV(\zeta).$$

Since $f \in C^{r-1}$ vanishes identically in \overline{D} , then $\partial^i f$ vanishes in ∂D for $i \leq [r] - 1$. Furthermore, by Taylor's theorem,

$$(4.11) \quad |\partial^i f(\zeta)| \leq C_i |f|_{D_\delta; [r]-1} d(\zeta)^{[r]-1-i}, \quad \zeta \in D_\delta, \quad 0 \leq i \leq [r] - 1.$$

Suppose that $[r] > 1$. Fix $z \in D$. Thus $|z - \zeta|^{-2j-2\mu_2}$ is C^∞ in $\zeta \in \mathbf{C}^n \setminus D$. Recall that $f_1(z, \zeta) = f(\zeta) A_1(W_1(z, \zeta), z, \zeta)$. Using (4.9) and (4.11), a straightforward computation shows that

$$(4.12) \quad |\partial_\zeta^j ((u^{-1} h_1)(z, \zeta))| \leq C_j(|W|_1) |f|_{D_\delta; [r]-1} d(\zeta)^{[r]-1-j},$$

for $z \in D$ and $\zeta \in D_\delta \setminus \overline{D}$ and $j \in \mathbf{N}$. Here $C_{i,j}(z) < \infty$. In particular, this allows us to apply the integration by parts to transform $K_1 f$.

Fix $z \in D$. When $[r] > 1$, we apply Stokes' theorem via Lemma 3.8 in which $S(\zeta) = u(z, \zeta)^{-1}h_1(z, \zeta)$, $m = [r] - 1$, $B(\zeta) = \Phi(z, \zeta)^{-(n-j+\mu_1)}$, and $b = 1$. Up to a constant multiple, we have

$$K_1 f(z) = \int_{\mathbb{C}^n \setminus \overline{D}} \frac{\partial_{\zeta_i^*} \{u(z, \zeta)^{-1}h_1(z, \zeta)\}}{\Phi^{n-j+\mu_1-1}(z, \zeta)} dV(\zeta), \quad \forall z \in D.$$

We also have

$$\begin{aligned} |\partial_{\zeta}^i \{u(z, \zeta)^{-1}(\partial_{\zeta_i^*} \circ u(z, \zeta)^{-1})^\ell \{h_1(z, \zeta)\}\}| &\leq C_{i+\ell}(|W|_1)|f|_{D_{\delta};[r]-1}d(\zeta)^{[r]-1-\ell-i}, \\ |\partial_{\zeta}^i \Phi^{-(n-j+\mu_1)+\ell}| &\leq C_{i+\ell}(|W|_1)(1+d(\zeta)^{1-i}). \end{aligned}$$

If $\mu_1 - \ell > 0$, we have $[r] - 1 - \ell > [r] - 1 - \mu_1 \geq 0$ by (4.8). Applying the integration by parts μ_1 times via Lemma 3.8 till $\mu_1 - \ell = 0$, we obtain

$$K_1 f(x) := \int_{\mathbb{C}^n \setminus \overline{D}} \frac{h_2(z, \zeta)}{\Phi^{n-j}(z, \zeta)} dV(\zeta), \quad \forall z \in D$$

with

$$h_2(z, \zeta) := (\partial_{\zeta_i^*} \circ u(z, \zeta)^{-1})^{\mu_1} \{h_1(z, \zeta)\}.$$

By the product and quotient rules, we can write $h_2(z, \zeta)$ as a linear combination of

$$(4.13) \quad h_3(z, \zeta) = f_2(z, \zeta) \tilde{N}_\lambda(\zeta - z),$$

where f_2 and \tilde{N}_λ have the form

$$(4.14) \quad \tilde{N}_\lambda(\zeta - z) = \frac{N_{1-\mu_0+\mu_2-\nu_1+\nu_2}(\zeta - z)}{|\zeta - z|^{2j+2\mu_2+2\nu_2}},$$

$$(4.15) \quad f_2(z, \zeta) = A_2(\widetilde{W}_1(z, \zeta), z, \zeta) \partial_{\zeta}^{\nu_i} \widetilde{W}_1(z, \zeta) \cdots \partial_{\zeta}^{\nu_4} \widetilde{W}_1(z, \zeta) \partial_{\zeta}^{\nu_3} f.$$

Furthermore, each \widetilde{W}_1 is one of $\hat{\partial}_{\zeta} W(z, \zeta)$, $\partial_z^{k_0} \hat{\partial}_{\zeta} W(z, \zeta)$, $u_1^{-1}(z, \zeta)$. And

$$(4.16) \quad \nu_1 + \cdots + \nu_i \leq \mu_1, \quad 1 - \mu_0 + \mu_2 - \nu_1 + \nu_2 \geq 0,$$

$$(4.17) \quad \lambda := (1 - \mu_0 + \mu_2 - \nu_1 + \nu_2) - (2j + 2\mu_2 + 2\nu_2).$$

We have proved that $\partial_z^{k-1} Kf$ is a linear combination of $K_1 f$, while $K_1 f$ is a linear combination of

$$(4.18) \quad K_2 f(z) = \int_{\mathbb{C}^n \setminus \overline{D}} f_2(z, \zeta) \frac{\tilde{N}_\lambda(\zeta - z)}{\Phi^{n-j}(z, \zeta)} dV(\zeta).$$

Since $f(\zeta) = 0$ in \overline{D} , it is easy to see that $K_2 f \in \mathcal{C}^\infty(D)$.

We want to estimate $K_2 f(z)$ in terms of $d(z)$. To achieve an estimate that has the form (4.18), we need to count the total numbers of derivatives in the expression of f_2 . In (4.15), we already apply $\nu_4 + \cdots + \nu_i$ extra derivatives on \widetilde{W} . Set

$$\lambda' = \nu_4 + \cdots + \nu_i.$$

Then $|\partial_{\zeta}^{\nu_3} f| \leq C|f|_{r-1}d(\zeta)^{r-1-\nu_3}$. Since $|\zeta - z| \geq d(\zeta)$ and $[r] - 1 - \nu_3 \geq 0$ by (4.8) and (4.16), we obtain for $\zeta \in \overline{D}^c$,

$$|\partial_{\zeta}^{\nu_3} f| \leq C|f|_{r-1}d(\zeta)^{r-1-\nu_3}, \quad |\partial_{\zeta}^{\nu_i} \widetilde{W}_1(z, \zeta) \cdots \partial_{\zeta}^{\nu_4} \widetilde{W}_1(z, \zeta)| \leq C(|W|_1)d(\zeta)^{-\lambda'},$$

where the last inequality follows from (4.9). Hence their z -derivatives can be estimated by $|W|_{D_\delta,1}$. Then we have proved that for $z \in D$ and $\zeta \notin \overline{D}$

$$(4.19) \quad \begin{aligned} |f_2(z, \zeta)| &\leq C(|W|_1)|f|_{r-1}d(\zeta)^{r-1-\lambda'-\nu_3} \\ &\leq C(|W|_1)|f|_{r-1}d(\zeta)^\alpha|\zeta - z|^{[r]-1-\nu_3-\lambda'} \end{aligned}$$

by using $[r] - 1 - \nu_3 - \lambda' \geq 0$ and $d(\zeta) \leq |\zeta - z|$. By the definition of \tilde{N}_λ , we have

$$|\tilde{N}_\lambda(\zeta - z)| \leq |\zeta - z|^\lambda, \quad |h_2(z, \zeta)| \leq C(|W|_1)|f|_{r-1}d(\zeta)^\alpha|\zeta - z|^{[r]-1-\nu_3-\lambda'+\lambda}.$$

We have just estimated h_2 . Since $f(\zeta)$ does not depend on z , the z -derivatives of $f_2(z, \zeta)$, given by (4.15), by using

$$\begin{aligned} |\partial_z^\ell f_2(z, \zeta)| &\leq C_\ell |f|_{r-1} d(\zeta)^\alpha |\zeta - z|^{[r]-1-\nu_3-\lambda'}, \quad [r] - 1 - \nu_3 - \lambda' \geq 0, \\ |\partial_z^\ell \tilde{N}_\lambda(\zeta - z)| &\leq C_\ell |\zeta - z|^{\lambda-\ell}, \\ |\partial_z^\ell \Phi^{-(n-j)}(z, \zeta)| &\leq C_\ell (|W|_1) |\Phi^{-(n-j)-\ell}(z, \zeta)|. \end{aligned}$$

We estimate \tilde{N}_λ first. We have

$$\begin{aligned} \lambda &= (1 - \mu_0 + \mu_2 - \nu_1 + \nu_2) - (2j + 2\mu_2 + 2\nu_2) \\ &= 1 - 2j - \mu_0 - \mu_2 - \nu_1 - \nu_2 \geq 1 - 2j - \mu_0 - \mu_2 - \mu_1 + \nu_3 + \lambda' \\ &\geq 2 - 2j - [r] + \nu_3 + \lambda' \end{aligned}$$

by the first inequalities in (4.16) and (4.8). Then for h_3 given by (4.13), we have

$$(4.20) \quad |\partial_z^\ell h_3(z, \zeta)| \leq C(|W|_1)|f|_{r-1}d(\zeta)^\alpha|\zeta - z|^{1-2j-\ell}.$$

We have expressed $\partial^{[r]-1} Kf$ as a linear combination of $K_2 f$ by exhausting all derivatives of f . Let $z \in D$. We want to show that

$$(4.21) \quad |\partial_z^2 K_2 f(z)| \leq C(|W|_1)|f|_{r-1}d(z)^{-1+(\alpha+1/2)}, \quad \alpha + 1/2 < 1,$$

$$(4.22) \quad |\partial_z^3 K_2 f(z)| \leq C(|W|_1)|f|_{r-1}d(z)^{-1+(\alpha-1/2)}, \quad \alpha + 1/2 \geq 1.$$

For $\ell = 2, 3$ we compute $\partial_z^\ell K_2 f$ by differentiating the integrand directly. The $\partial_z^2 K_2 f$ is a sum of three kinds of terms

$$\begin{aligned} J_0 f(z) &= \int_{\mathbf{C}^n \setminus \overline{D}} \frac{f_2(z, \zeta)}{\Phi^{n-j}(z, \zeta)} \partial_z^2 \left\{ \tilde{N}_\lambda(\zeta - z) \right\} dV(\zeta), \\ J_1 f(z) &= \int_{\mathbf{C}^n \setminus \overline{D}} \frac{f_2(z, \zeta)}{\Phi^{n-j+1}(z, \zeta)} \partial_z \left\{ \tilde{N}_\lambda(\zeta - z) \right\} dV(\zeta), \\ J_2 f(z) &= \int_{\mathbf{C}^n \setminus \overline{D}} \frac{f_2(z, \zeta)}{\Phi^{n-j+2}(z, \zeta)} \left\{ \tilde{N}_\lambda(\zeta - z) \right\} dV(\zeta), \end{aligned}$$

where f_2 still has the form (4.15) while ν_4, \dots, ν_i are unchanged. Therefore we obtain

$$\begin{aligned} |J_0 f(z)| &\leq C(|W|_1)|f|_{r-1} \int_{\mathcal{U} \setminus \overline{D}} \frac{d(\zeta)^\alpha}{|\Phi(z, \zeta)|^{n-j} |\zeta - z|^{1+2j}} dV(\zeta), \\ |J_1 f(z)| &\leq C(|W|_1)|f|_{r-1} \int_{\mathcal{U} \setminus \overline{D}} \frac{d(\zeta)^\alpha}{|\Phi(z, \zeta)|^{n-j+1} |\zeta - z|^{2j}} dV(\zeta), \\ |J_2 f(z)| &\leq C(|W|_1)|f|_{r-1} \int_{\mathcal{U} \setminus \overline{D}} \frac{d(\zeta)^\alpha}{|\Phi(z, \zeta)|^{n-j+2} |\zeta - z|^{-1+2j}} dV(\zeta). \end{aligned}$$

Recall that $1 \leq j < n$. For $z \in D$ and $\zeta \notin D$, we have $C'|\zeta - z| \geq |\Phi(z, \zeta)| \geq C|\zeta - z|^2$. Thus it suffices to estimate the last integral for $j = n - 1$. Set

$$\widehat{J}_2(z) := \int_{\mathcal{U} \setminus \overline{D}} \frac{d(\zeta)^\alpha}{|\Phi(z, \zeta)|^3 |\zeta - z|^{2n-3}} dV(\zeta).$$

Fix $\zeta_0 \in \partial D$ and a small neighborhood ω_0 of ζ_0 . Let $z \in \omega_0 \cap D$ and $\zeta \in \omega_0 \setminus D$. Note that $r(\zeta) \approx \text{dist}(\zeta, \partial D) = d(\zeta)$. We now use the assumptions that

$$|\Phi(z, \zeta)| \geq c_*(d(z) + s_1 + |s_2(\zeta)| + |t(\zeta)|^2), \quad |\zeta - z| \geq c_*|t(\zeta)|.$$

We also have

$$d(\zeta)/C \leq r(\zeta) = s_1 \leq |(s_1, s_2)(\zeta)|, \quad \zeta \in D_\delta \setminus \overline{D}.$$

Using polar coordinates for $(s_1, s_2) \in \mathbf{R}^2$ and $(t_3, \dots, t_{2n}) \in \mathbf{R}^{2n-2}$, we obtain for $z \in \omega_0$

$$\widehat{J}_2(z) \leq C \int_{s=1}^1 \int_{t=0}^1 \frac{s^{\alpha+1} ds dt}{(d(z) + s + t^2)^3},$$

which is less than $Cd(z)^{\alpha-1}$ by Lemma 4.3 in which we take $\beta = 0$ and $0 \leq \alpha < 1/2$. We have verified (4.21).

Consider now the case $1/2 < \alpha < 1$ for a). This requires us to estimate $\partial_z^3 K_2 f$. This is a sum of terms

$$\widetilde{J}_i f(z) := \int_{\mathbf{C}^n \setminus \overline{D}} \frac{f_2(z, \zeta)}{\Phi^{n-j+i}(z, \zeta)} \partial_z^{3-i} \left\{ \widetilde{N}_\lambda(\zeta - z) \right\} dV(\zeta)$$

for $i = 0, 1, 2, 3$. The worst term is $\widetilde{J}_3 f(z)$ with $j = n - 1$ and $i = 3$. We have

$$|\widetilde{J}_3 f(z)| \leq C'(|W|_1)|f|_{r-1} \int_{s=1}^1 \int_{t=0}^1 \frac{s^{\alpha+1} ds dt}{(d(z) + s + t^2)^4},$$

which is less than $C|f|_{r-1} d(z)^{\alpha-3/2}$ by Lemma 4.3 with $\beta = 1$ and $1/2 < \alpha < 3/2$.

(ii). **Case 1**, $\alpha \neq 0, 1/2$. Recall that Kf is C^∞ in D and its derivatives on a compact subset of D can be estimated easily by the sup norm of f . When $\alpha \neq 0, 1/2$, by the Hardy-Littlewood lemma for Hölder spaces we get the estimate in (ii) from (i) immediately. The same argument by the Hardy-Littlewood also gives us $|Kf|_{\overline{D}, k+1/2} \leq C_k |f|_{\mathcal{U}, k-1}$ when k is a positive integer, which is a weaker version of (ii) for the $\Lambda_{k+1/2}$ estimate.

Case 2, $\alpha = 1/2$. For the case $\alpha = 1/2$, one can use a version of Hardy-Littlewood lemma to conclude that $K_2 f \in \Lambda_1$; see [39]. We will however provide a more direct proof for the case $\alpha = 1/2$, by using the estimates in (i).

In fact we will provide an argument that actually works for $0 < \alpha < 1$. Let us show

$$|K_2 f|_{\Lambda_{r+3/2}} \leq C_r (|W|_1) |f|_{\Lambda_r}.$$

We may assume that the f vanishes when $|(s_1, s_2)| > 1$ or $|t| > 1$. We consider a dyadic decomposition $K_2 f = \sum_{k \geq 1} g_k$ with

$$g_k(z) = \int_{(s_1, s_2) \in A_k^+} f_2(z, \zeta) \frac{\tilde{N}_\lambda(\zeta - z)}{\Phi^{n-j}(z, \zeta)} dV(\zeta), \quad z \in D$$

for $A_k^+ = \{(s_1, s_2) : 2^{-k} \leq |(s_1, s_2)| < 2^{-k+1}, s_1 \geq 0\}$. By (4.13) and (4.20) we still have $|h_3(z, \zeta)| \leq |f|_{r-1} s_1^\alpha \leq |f|_{r-1} |(s_1, s_2)|^\alpha$. Hence

$$\begin{aligned} |\partial^i h_k(z)| &\leq \int_{A_k^+} \int_{|t| < 1} \frac{C'(|W|_1) |f|_{r-1} s_1^{\alpha+1}}{(s_1 + |s_2| + |t|^2)^{1+i} (s_1 + |s_2| + |t|)^{2n-3}} ds_1 ds_2 dt \\ &\leq C(|W|_1) |f|_{r-1} \int_{s=2^{-k}}^{2^{-k+1}} \int_{t=0}^1 \frac{s^{\alpha+1}}{(s + |t|^2)^{1+i}} ds dt \leq C'(|W|_1) |f|_{r-1} 2^{-k(\alpha-i+3/2)}. \end{aligned}$$

Now (ii) for $0 < \alpha < 1$ follows from Lemmas 3.11 and 3.10.

Case 3, $r > 1$ is an integer. We will achieve the $\Lambda_{r+1/2}$ estimate by the real interpolation theory. Fix $d\bar{z}^I$ with $|I| = q > 0$ and fix $d\bar{z}^J$ with $|J| = q-1$. Let $\{\psi\}_J$ denote the coefficients of $d\bar{z}^J$ for a $(0, q-1)$ form ψ . Consider the linear mapping

$$L_J: f \rightarrow \left\{ \int_{\mathcal{U} \setminus \overline{D}} \Omega_{0,q}^{01} \wedge [\bar{\partial}, E](f d\bar{z}^I) \right\}_J.$$

Assume that $k \geq 2$ be an integer. Let E be the linear extension operator for functions defined in \overline{D} , given in Lemma 3.11. Using (4.22), we have

$$|EL_J f|_{\mathbf{C}^n; k-\epsilon+\frac{1}{2}} \leq C_1 |L_J f|_{\mathbf{C}^n; k-\epsilon+\frac{1}{2}} \leq C_1 C_k (|W|_1) |f|_{\mathbf{C}^n; k-1-\epsilon}.$$

Using (4.21), we have $|EL_J f|_{\mathbf{C}^n; k+\epsilon+\frac{1}{2}} \leq C_1 C_{k+1} (|W|_1) |f|_{\mathbf{C}^n; k-1+\epsilon}$. The estimate follows from interpolation via Proposition 3.12. The assertion (ii) is proved. \square

Remark 4.5. We can approximate $\varphi \in \Lambda_1(\mathbf{C}^n)$ by bounded C^1 forms φ_j in \mathbf{C}^n which converges in the sup norm to φ . However, we do not have a useful limit u of $H_q \varphi_j$ as $j \rightarrow \infty$, in order to conclude that $\bar{\partial}u = \varphi$ when φ is $\bar{\partial}$ closed.

We now turn to the estimate of holomorphic projection H_0 . The analogous estimate for the boundary operator in (2.5) is in Ahern-Schneider [1], where $\partial D \in C^\infty$ is used. We need to restrict to $r > 1$, requiring $\partial D \in C^2$ only.

Lemma 4.6. *Let $0 \leq \alpha < 1$, $0 < \delta < 1/2$, and $n \geq 2$. Then*

$$\int_0^1 \int_0^1 \frac{s^{\alpha+1} t^{2n-3}}{(\delta + s + t^2)^{n+2}} dt ds \leq \frac{C_n}{1-\alpha} \delta^{\alpha-1}.$$

Proof. We estimate the integrals I of the integrand by a covering of $[0, 1] \times [0, 1]$:

(i) $\delta \leq t^2 \leq s$.

$$I \leq \int_\delta^1 \int_{t=0}^{\sqrt{s}} \frac{s^{\alpha+1} t^{2n-3}}{s^{n+2}} dt ds \leq \int_\delta^1 s^{\alpha-2} ds \leq C \delta^{\alpha-1}.$$

(ii) $\delta \leq s \leq t^2$.

$$I \leq \int_{\sqrt{\delta}}^1 \int_{s=0}^{t^2} \frac{s^{\alpha+1} t^{2n-3}}{t^{2n+4}} ds dt \leq \int_{\sqrt{\delta}}^1 t^{2\alpha-3} dt \leq C \delta^{\alpha-1}.$$

(iii) $t^2 \leq \delta \leq s$.

$$I \leq \int_0^{\sqrt{\delta}} \int_{s=\delta}^1 \frac{s^{\alpha+1} t^{2n-3}}{s^{n+2}} dt ds \leq \delta^{\alpha-n} \delta^{n-1} \leq C\delta^{\alpha-1}.$$

(iv) $s \leq \delta \leq t^2$.

$$I \leq \int_0^\delta \int_{t=\sqrt{\delta}}^1 \frac{s^{\alpha+1} t^{2n-3}}{t^{2n+4}} dt ds \leq \delta^{\alpha+2} \delta^{-3} \leq C\delta^{\alpha-1}.$$

(v) $t^2 \leq s \leq \delta$.

$$I \leq \int_0^\delta \int_{t=0}^{\sqrt{s}} \frac{s^{\alpha+1} t^{2n-3}}{\delta^{n+2}} ds dt \leq \delta^{-n-2} \int_\delta^1 s^{n+\alpha} ds \leq C\delta^{\alpha-1}.$$

(vi) $\delta \leq t^2 \leq s$.

$$I \leq \int_\delta^1 \int_{t=0}^{\sqrt{s}} \frac{s^{\alpha+1} t^{2n-3}}{s^{n+2}} dt ds \leq C\delta^{\alpha-1}. \quad \square$$

Proposition 4.7. *Let $r > 1$. Let D, Φ, g^1 be as in Proposition 4.4. Suppose that $f \in C^1(\mathbf{C}^n)$ is a function vanishing in D . Then*

$$\begin{aligned} \|H_0 f\|_{\Lambda_r(\overline{D})} &\leq C_r(|W|_1) \|f\|_{\Lambda_r(\mathcal{U} \setminus D)}, \quad r > 1, \\ |\partial_z^2 H_0 f(z)| &\leq C_1(|W|_1) \text{dist}(z, \partial D)^{-1} |f|_{1, \mathcal{U} \setminus D}, \quad z \in D. \end{aligned}$$

Proof. Let $k = [r] \geq 1$. We first consider the case $f \in C^r(\overline{D})$. The above proof for $H_i f$ with $i > 0$ can be adapted easily. Let $\partial_z^{k+1} H_0 f$ be a $(k+1)$ -th order derivative of $H_0 f$. It is a linear combination of

$$Kf(z) = \int_{\mathcal{U} \setminus \overline{D}} f(\zeta) \partial_z^{k+1} \left\{ \frac{A(W_1(z, \zeta), z, \zeta)}{\Phi^n(z, \zeta)} \right\} dV(\zeta).$$

Let $z_0 \in \partial D$. Using a partition of unity, we may assume that for a neighborhood B_0 of z_0 in \mathbf{C}^n and for some j , we have

$$\text{supp } f \subset B_0 \setminus D; \quad u(z, \zeta) := \partial_{\zeta_j} \Phi(z, \zeta) \neq 0, \quad z, \zeta \in B_0.$$

Applying integration by parts $k-1$ times, we write Kf as a linear combination of $K_1 f$ with

$$K_1 f(x) := \int_{\mathbf{C}^n \setminus \overline{D}} \frac{f_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta), \quad \forall z \in D$$

with $f_1(z, \zeta) = A(W_1(z, \zeta), z, \zeta) \partial_{\zeta}^{\nu_\ell} W_1 \cdots \partial_{\zeta}^{\nu_1} W_1 \partial_{\zeta}^{\nu_0} f$ and

$$\nu_0 + \cdots + \nu_\ell = k-1.$$

Since $f(\zeta) = 0$ in \overline{D} , it is easy to see that $K_1 f \in \mathcal{C}^\infty(D)$. We have for $\alpha > 0$

$$|f_2(z)| \leq C(|W|_1) |f|_r \int_{s=0}^1 \int_{t=0}^1 \frac{s^{\alpha+1} t^{2n-3}}{(d(z) + s + t^2)^{n+2}} ds dt \leq C'(|W|_1) |f|_r d(z)^{\alpha-1}.$$

When r is a positive integer, the estimate follows from interpolation by Proposition 3.12. \square

5. REGULARIZED HENKIN-RAMÍREZ FUNCTIONS

We now discuss our result for strictly pseudoconvex domains. We first strengthen the following result on the classical Henkin-Ramírez functions.

Proposition 5.1. *Let D be a bounded strictly pseudoconvex domain in \mathbf{C}^n with C^2 boundary. Suppose that $\rho_0 \in C^2(\mathcal{U})$, $\partial D = \{z \in \mathcal{U} : \rho_0 = 0\}$, and $\partial\rho_0 \neq 0$ in ∂D . Let $D_\delta = \{z \in \mathbf{C}^n : \text{dist}(z, D) < \delta\}$ and $D_{-\delta} = \{z \in D : \text{dist}(z, \partial D) > \delta\}$. Let $\rho = E_2(e^{L_0\rho_0} - 1)$ be a regularized C^2 defining function of D , where $L_0 > 0$ is sufficiently large so that $e^{L_0\rho_0} - 1$ is strictly plurisubharmonic in a neighborhood ω of ∂D . There exist $\delta > 0$ and functions W satisfying the following.*

- (i) W is defined in $D_\delta \times (D_\delta \setminus D_{-\delta})$, $\Phi(z, \zeta) = W(z, \zeta) \cdot (\zeta - z) \neq 0$ for $\rho(z) \leq \rho(\zeta)$ and $\zeta \neq z$, $W(\cdot, \zeta)$ is holomorphic in D_δ for $z \in D_\delta$, and $W \in C^1(D_\delta \times (D_\delta \setminus D_{-\delta}))$.
- (ii) If $|\zeta - z| < \epsilon$ and $\zeta \in D_\delta \setminus D_{-\delta}$, then $\Phi(z, \zeta) = F(z, \zeta)M(z, \zeta)$, $M(z, \zeta) \neq 0$ and

$$F(z, \zeta) = - \sum \frac{\partial r}{\partial \zeta_j} (z_j - \zeta_j) + \sum a_{jk}(\zeta) (z_j - \zeta_j)(z_k - \zeta_k),$$

$$\text{Re } F(z, \zeta) \geq \rho(\zeta) - \rho(z) + |\zeta - z|^2/C,$$

with $(M, F) \in C^1(D_\delta \times (D_\delta \setminus D_{-\delta}))$ and $a_{jk} \in C^\infty(\mathbf{C}^n)$.

- (iii) For $z \in D_\delta$ and $\zeta \in D_\delta \setminus \overline{D}$, we have

$$(5.1) \quad |\partial_z^i \partial_\zeta^j W(z, \zeta)| \leq C_{i,j}(D, L_0, |\rho_0|_{\overline{D}, 2}) \sum_{j_1+j_2=j} \delta^{-i-j_1} \left\{ 1 + d(\zeta, \partial D)^{1-j_2} \right\}.$$

The functions W will be called regularized Henkin-Ramírez functions.

Proof. When ρ is strictly plurisubharmonic, the proof for (i) and (ii) is in Øvrelid [41] and see also Henkin-Leiterer [26, Thm. 2.4.3, p. 78; Thm. 2.5.5, p. 81], and Range [44, Prop. 3.1, p. 284]. Therefore, only (iii) is new.

Fix δ_0 so that $\overline{D_{\delta_0}} \setminus D$ is contained in $\mathcal{U} \cap \omega$. We have

$$\delta_1 = \min\{\rho(\zeta) : \zeta \in \partial D_{\delta_0}\} > \delta_0/C.$$

We have

$$\sum_{j,k} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k} t_j \bar{t}_k \geq C_0 |t|^2, \quad \zeta \in \omega,$$

with $C_0 > 0$. Define $D_c^* = \{z \in D_{\delta_0} : \rho(z) < c\}$. We take

$$(5.2) \quad F(z, \zeta) := - \sum \frac{\partial \rho}{\partial \zeta_j} (z_j - \zeta_j) - \sum a_{ij}(\zeta) (z_i - \zeta_i)(z_j - \zeta_j),$$

where $a_{jk} \in C^\infty(\mathbf{C}^n)$ with $|a_{jk}(\zeta) - \frac{\partial^2}{\partial \zeta_j \partial \zeta_k} \rho| < 1/C$ for $\zeta \in \mathcal{U}$.

Fix ϵ sufficiently small so that for $|\zeta - z| < \epsilon$ and $\zeta, z \in D_{\delta_0} \setminus D_{-\delta_0}$,

$$\text{Re } F(z, \zeta) \geq r(\zeta) - r(z) + |z - \zeta|^2/C_0.$$

Let χ be a C^∞ function satisfying $\chi(\zeta) = 1$ for $|\zeta| < 3\epsilon/4$ and $\chi(\zeta) = 0$ for $|\zeta| > 7\epsilon/8$. Take $\delta_2 < \frac{1}{4C_0}(\frac{3}{4}\epsilon)^2$ and $\delta_2 \in (0, \delta_1)$. For $z \in D_{\delta_2}^*$, $\zeta \in D_{\delta_1}^* \setminus D_{-\delta_2}^*$, and $|\zeta - z| > 3\epsilon/4$, we have

$$\text{Re } F(z, \zeta) \geq r(\zeta) - r(z) + |z - \zeta|^2/C_0 \geq -2\delta_2 + \frac{1}{C_0} \left(\frac{3}{4}\epsilon \right)^2 > \frac{1}{2C_0} \left(\frac{3}{4}\epsilon \right)^2.$$

Thus we can define

$$(5.3) \quad f(z, \zeta) = \begin{cases} \bar{\partial}_z(\chi(\zeta - z) \log F(z, \zeta)) & \text{if } 3\epsilon/4 < |\zeta - z| < 7\epsilon/8, \\ 0 & \text{otherwise,} \end{cases}$$

for $z \in D_{\delta_2}^*$ and $\zeta \in D_{\delta_1}^* \setminus D_{-\delta_2}^*$. Define

$$(5.4) \quad u(z, \zeta) = T_{D_{\delta_2}^*} f(\cdot, \zeta)(z), \quad \forall z \in D_{\delta_2}^*, \quad \zeta \in D_{\delta_1}^* \setminus D_{-\delta_2}^*.$$

Here $T_0 = T_{D_{\delta_2}^*}$ is a linear $\bar{\partial}$ solution operator that admits an interior super-norm estimate on $D_{\delta_2}^*$ (see [26, Thm. 2.3.5, p. 76]). Namely, for any $\bar{\partial}$ -closed $(0, 1)$ form φ in $D_{\delta_2}^*$, $\bar{\partial}T_0\varphi = \varphi$ and

$$(5.5) \quad |T_0\varphi|_{L^\infty(D_{\delta_3}^*)} \leq C_0^* |\varphi|_{L^\infty(D_{\delta_2}^*)},$$

for $\delta_3 \in (0, \delta_2)$. By the linearity of T_0 and Proposition 2.2, the $u(z, \zeta)$ and $\partial_\zeta u(z, \zeta)$ are uniformly continuous in $D_{\delta_2}^* \times (D_{\delta_1}^* \setminus D_{-\delta_2}^*)$. We also have

$$|f(z, \cdot)|_{D_{\delta_1}^* \setminus D_{-\delta_2}^*, 1} \leq C(|\rho|_2), \quad z \in D_{\delta_2}^*, \quad |u(z, \cdot)|_{D_{\delta_1}^* \setminus D_{-\delta_2}^*, 1} \leq C_0^* C(|\rho|_2), \quad z \in D_{\delta_3}^*.$$

Define for $z \in D_{\delta_3}^*$ and $\zeta \in D_{\delta_1}^* \setminus D_{-\delta_2}^*$

$$(5.6) \quad \Phi(z, \zeta) := \begin{cases} F(z, \zeta) e^{-u(z, \zeta)} & \text{if } |\zeta - z| \leq 3\epsilon/4, \\ e^{\chi(\zeta - z) \log F(z, \zeta) - u(z, \zeta)} & \text{otherwise.} \end{cases}$$

Then $|\Phi(\cdot, \zeta)|_{D_{\delta_3}^*, 0} \leq C|\rho|_2$ for $\zeta \in D_{\delta_1}^* \setminus D_{-\delta_2}^*$. Also

$$|\Phi(z, \cdot)|_{D_{\delta_1}^* \setminus D_{-\delta_2}^*, 1} \leq C(C_*, |\rho|_2), \quad z \in D_{\delta_3}^*.$$

Fix $\delta_4^* \in (0, \delta_3^*)$. By Hefer's decomposition theorem [26, p. 81], there are continuous linear mappings $T_j: \mathcal{O}(D_{\delta_3}^*) \rightarrow \mathcal{O}((D_{\delta_4}^*)^2)$ so that $h(\tilde{z}) - h(z) = \sum_{j=1}^n T_j h(z, \tilde{z})(\tilde{z}_j - z_j)$. Then we have

$$(5.7) \quad \Phi(\tilde{z}, \zeta) - \Phi(z, \zeta) = \sum T_j \Phi(\cdot, \zeta)(z, \tilde{z})(\tilde{z}_j - z_j).$$

Set $W_j(z, \zeta) = T_j \Phi(\cdot, \zeta)(z, \zeta)$. We know that $T_j \Phi(\cdot, \zeta)(z, \eta)$ is holomorphic in z, η . We express the boundedness of T_j as

$$(5.8) \quad |T_j h|_{L^\infty((D_{\delta_4}^*)^2)} \leq C_j^* |h|_{L^\infty(D_{\delta_3}^*)}, \quad h \in \mathcal{O}(D_{\delta_3}^*).$$

The linearity and continuity of T_j implies that W_j is continuous in (z, ζ) .

We now use the fact that ρ is a regularized C^2 defining function for the domain D to obtain pointwise estimates for the derivatives of $\Phi(z, \zeta)$ in

$$z \in D_{\delta_5}^*, \quad \zeta \in D_{\delta_4}^* \setminus \overline{D}.$$

Here we take $\delta_5^* \in (0, \delta_4^*)$. This allows us to use Cauchy inequality in the z variables.

By (5.3), (5.4) and the linear estimate (5.5), we first see that for each j , $\partial_\zeta^j u(z, \zeta)$ are continuous in $(z, \zeta) \in D_{\delta_3}^* \times (D_{\delta_1}^* \setminus D_{-\delta_2}^*)$. Moreover,

$$|\partial_\zeta^j u(\cdot, \zeta)|_{D_{\delta_3}^*, 0} \leq C_j(C_0^*, |\rho|_2)(1 + d(\zeta)^{1-j}).$$

Here we have use $|\partial_\zeta^j \rho(\zeta)| \leq C_j(1 + d(\zeta)^{1-j})$ as well as the product rule for

$$\log F(z, \zeta) = \log \operatorname{Re} F(z, \zeta) + \log \left(1 + i \frac{\operatorname{Im} F(z, \zeta)}{\operatorname{Re} F(z, \zeta)} \right), \quad 3\epsilon/4 < |\zeta - z| < 7\epsilon/8,$$

where $z \in D_{\delta_3}^*$, $\zeta \in D_{\delta_1}^* \setminus D_{-\delta_2}^*$. By (5.6), we get $|\partial_\zeta^j \Phi(\cdot, \zeta)|_{D_{\delta_3}^*, 0} \leq C_j(1 + d(\zeta)^{1-j})$. Here and in what follows, we let

$$C_j := C_j(C_0^*, \dots, C_n^*, |\rho|_2).$$

By the linearity and continuity of T_j and the holomorphicity of $T_j \Phi(\cdot, \zeta)(z, \eta)$ in η , we have

$$\partial_\zeta^\alpha W_\ell(z, \zeta) = \sum \binom{\alpha}{\beta} \partial_\eta^{\alpha-\beta} \Big|_{\eta=\zeta} T_\ell \partial_\zeta^\beta \Phi(\cdot, \zeta)(z, \eta)$$

for $z \in D_{\delta_4}^*$ and $\zeta \in D_{\delta_4}^* \setminus D_{-\delta_2}^*$. By the linearity of estimate (5.8) for T_ℓ and Cauchy inequality applied to the last term, we get $|\partial_\zeta^j W(\cdot, \zeta)|_{D_{\delta_5}^*, 0} \leq C_j \sum_{j_1+j_2=j} \text{dist}(D_{\delta_5}^*, \partial D_{\delta_4}^*)^{-j_1} (1 + d(\zeta)^{1-j_2})$ for $j = 1, 2, \dots$. By Cauchy inequality, we get

$$|\partial_\zeta^j W(\cdot, \zeta)|_{D_{\delta_5}^*, i} \leq C_j \sum_{j_1+j_2=j} \text{dist}(D_{\delta_5}^*, \partial D_{\delta_4}^*)^{-i-j_1} (1 + d(\zeta)^{1-j_2}).$$

for $\zeta \in D_{\delta_5}^* \setminus D_{-\delta_2}^*$. Finally, we fix $\delta \in (0, \delta_5)$. We have achieved (5.1). \square

Theorem 5.2. *Let $D = \{z \in \mathcal{U} : \rho_0 < 0\}$ be a strictly pseudoconvex domain with C^2 boundary that is relatively compact in \mathcal{U} and $\rho_0 \in C^2(\mathcal{U})$. Suppose that $d\rho_0 \neq 0$ in ∂D . Let H_q be defined by (2.9) and (2.10), where $g^1 = W$ is the Henkin-Ramírez function $D_\delta \times D_\delta \setminus D_{-\delta}$ as in Proposition 5.1 and $\Phi(z, \zeta) = W(z, \zeta) \cdot (\zeta - z)$. Let φ be a $(0, q)$ form such that $\varphi, \bar{\partial}\varphi$ are in $C^1(\bar{D})$. Then in D*

$$(5.9) \quad \varphi = \bar{\partial} H_q \varphi + H_{q+1} \bar{\partial} \varphi, \quad 1 \leq q \leq n,$$

$$(5.10) \quad \varphi_0 = H_0 \varphi_0 + H_1 \bar{\partial} \varphi_0.$$

Moreover, we have

$$(5.11) \quad \|H_q \varphi\|_{\Lambda_{r+1/2}(\bar{D})} \leq C_r(D) \|\varphi\|_{\Lambda_r(\bar{D})}, \quad r > 1, q > 0,$$

$$(5.12) \quad \|H_q \varphi\|_{D; 3/2} \leq C_1(D) \|\varphi\|_{D; 1}, \quad q > 0,$$

$$(5.13) \quad \|H_0 \varphi\|_{\Lambda_r(\bar{D})} \leq C_r(D) \|\varphi\|_{\Lambda_r(\bar{D})}, \quad r > 1,$$

$$(5.14) \quad \|\partial_z^2 H_0 \varphi(z)\| \leq C_1(D) \text{dist}(z, \partial D)^{-1} \|\varphi\|_{D; 1}, \quad z \in D.$$

The constants $C_r(D)$ are stable under a small C^2 perturbation. They depend on ϵ, N, M in Definition 3.6, L in Lemma 3.4, $|\partial \rho_0|_{D \setminus D_{-\delta_0}; 0}$, $|\partial \bar{\partial} \rho_0|_{D \setminus D_{-\delta_0}; 0}$, $|\partial \rho_0|_{\bar{D}; 2}$, as well as the constants $L_0, \delta, C_0^*, \dots, C_n^*$ in the proof of Proposition 5.1. Therefor, $C_r(\tilde{D}) \leq \hat{C}_r(D, \epsilon) < \infty$ for all $r \in (0, \infty)$ when \tilde{D} has a defining function $\tilde{\rho}$ such that $|\tilde{\rho}_0 - \rho_0|_{\mathcal{U}; 2} < \epsilon$ for sufficiently small ϵ .

Proof. Let $\rho = E_2(e^{L_0 \rho_0} - 1)$ as in Proposition 5.1. Let us first choose the local coordinates described in Proposition 4.4. As in [34], we take

$$s_1 = \rho(\zeta), \quad s_2 = \text{Im}(\rho_\zeta \cdot (\zeta - z)), \quad t = (\text{Re}(\zeta' - z'), \text{Im}(\zeta' - z')).$$

Let F be as in Proposition 5.1. First, we have $|F(z, \zeta)| \geq \text{Re}(F(z, \zeta)) \geq c_* |\zeta - z|^2$. Note that $\rho(\zeta) \approx d(\zeta)$ and $-\rho(z) \approx d(z)$. Then we have $2|F(\zeta, z)| \geq s_2 + \rho(\zeta) - \rho(z) + c_* |\zeta - z|^2 \geq c'_*(d(z) + s_1 + |s_2| + |t|^2)$. Since $z \in D$ and $\zeta \notin \bar{D}$, we also have $d(z) \leq |\zeta - z|$, $d(\zeta) \leq |\zeta - z|$, $\rho(\zeta) \approx d(\zeta) \leq |\zeta - z|$. Hence $d(z) + d(\zeta) + s_1 + |s_2| + |t| \leq C|\zeta - z|$. Thus, we have proved verified (4.4)-(4.5). We obtain the desired estimates by Proposition 5.1, Proposition 2.2, Proposition 4.4 (ii), and Proposition 4.7. \square

6. BOUNDARY REGULARITIES OF THE ELLIPTIC DIFFERENTIAL COMPLEX FOR THE LEVI-FLAT EUCLIDEAN SPACE

We consider the complex for the exterior differential $\mathcal{D} := d_t + \bar{\partial}_z$ for $(z, t) \in \mathbf{C}^n \times \mathbf{R}^M$. We will also write it as $\mathcal{D} = d^0 + \bar{\partial}$.

The Poincaré lemma for a q -form on a bounded star-shaped domain S has the form

$$\phi = d^0 R_q \phi + R_{q+1} d^0 \phi, \quad q > 0; \quad \phi = R_1 d^0 \phi + \phi(0), \quad q = 0;$$

$$R_q \phi(t) := \int_{\theta \in [0,1]} H^* \phi(t, \theta),$$

where $H(t, \theta) = \theta t$ for $(t, \theta) \in S \times [0, 1]$; see [48, p. 224], [50, p. 105]. If $\phi(t) = f(t) dt_1 \wedge \cdots \wedge dt_q$, we have $R_q \phi(t) = \left\{ \int_0^1 f(\theta t) \theta^{q-1} d\theta \right\} \sum (-1)^{j-1} t_j dt_1 \cdots \widehat{dt_j} \cdots \wedge dt_q$. It is immediate that for $q > 0$

$$(6.1) \quad |R_q \phi|_{S;r} \leq C_r |\phi|_{S;r}, \quad 0 \leq r < \infty.$$

Here C_r depends only on the diameter of S . By the interpolation argument, we obtain

$$(6.2) \quad |R_q \phi|_{\Lambda_r(\bar{S})} \leq C_r(D) \rho |\phi|_{\Lambda_r(\bar{S})}, \quad 0 < r < \infty,$$

provided S is a bounded Lipschitz domain.

A differential form φ is called of *mixed type* $(0, q)$ if $\varphi = \sum_{i=0}^q [\varphi]_i$, where

$$[\varphi]_i = \sum_{|I|=i, |I|+|J|=q} a_{IJ} d\bar{z}^I \wedge dt^J.$$

Thus $[\varphi]_i = 0$, for $i > n$. The \mathcal{D} acts on a function f and a $(0, q)$ form as follows

$$\mathcal{D}f = \sum \frac{\partial f}{\partial t_m} dt_m + \sum \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha, \quad \mathcal{D} \sum a_{IJ} d\bar{z}^I \wedge dt^J = \sum \mathcal{D}a_{IJ} \wedge d\bar{z}^I \wedge dt^J.$$

We have $\mathcal{D}^2 = 0$ and $d^0 \bar{\partial} + \bar{\partial} d^0 = 0$. We also have

$$[\mathcal{D}\varphi]_0 = d^0 [\varphi]_0, \quad [\mathcal{D}\varphi]_i = d^0 [\varphi]_{i+1} + \bar{\partial} [\varphi]_i, \quad 0 < i \leq n.$$

For $\varphi = \sum \varphi_{IJ} d\bar{z}^I \wedge dt^J = \sum \tilde{\varphi}_{IJ} dt^J \wedge d\bar{z}^I$ on $\bar{D} \times \bar{S}$, define

$$H_i \varphi = \sum_{|I|=i} H_i(\varphi_{IJ} d\bar{z}^I) \wedge dt^J, \quad R_i \varphi = \sum_{|J|=i} R_i(\tilde{\varphi}_{IJ} dt^J) \wedge d\bar{z}^I.$$

Thus $H_i \varphi = H_i [\varphi]_i$, while $R_{q-i} \varphi = R_{q-i} [\varphi]_i$ if φ has the (mixed) type $(0, q)$.

Definition 6.1. Let $0 < r \leq 1$ and $0 \leq \alpha < 1$. Let $\Lambda_*^{r,0}(\bar{D} \times \bar{S})$ be the set of continuous functions f in $\bar{D} \times \bar{S}$ so that $|f(\cdot, t)|_{\Lambda_r(\bar{D})}$ are bounded in $t \in \bar{S}$. For $a > 1$ and $k \in \mathbf{N}$, let $\Lambda_*^{a,k}(\bar{D} \times \bar{S})$ be the set of functions f so that $\partial_z^i \partial_t^j f$ are in $\Lambda_*^{a-i,0}(\bar{D} \times \bar{S})$ for $j \leq k$ and $i < a$. Define $C_*^{a,k}(\bar{D} \times \bar{S})$ analogously.

We now derive the following homotopy formulae.

Proposition 6.2. *Let $1 \leq q \leq n + m$. Let D be a bounded strictly pseudoconvex domain in \mathbf{C}^n with $\partial D \in C^2$ and let S be a bounded domain in \mathbf{R}^m so that $\theta S \subset S$ for $\theta \in [0, 1]$. Let φ be a mixed $(0, q)$ form in $D \times S$.*

(i) If $\varphi \in C_*^{1,1}(\overline{D} \times \overline{S})$ and $\overline{\partial}\varphi$ are in $C_*^{1,0}(\overline{D} \times \overline{S})$, then

$$(6.3) \quad \varphi = \mathcal{D}T_q\varphi + T_{q+1}\mathcal{D}\varphi,$$

$$(6.4) \quad T_q\varphi = R_qH_0[\varphi]_0 + \sum_{i>0} H_i[\varphi]_i.$$

(ii) If $\varphi \in C^1(\overline{D} \times \overline{S})$ and $\overline{\partial}[\varphi]_q(\cdot, 0) \in C^1(\overline{D})$, then

$$(6.5) \quad \varphi = \mathcal{D}\tilde{T}_q\varphi + \tilde{T}_{q+1}\mathcal{D}\varphi,$$

$$(6.6) \quad \tilde{T}_q\varphi(z, t) = H_q[\varphi]_q(\cdot, 0)(z) + \sum_{i<q} R_{q-i}[\varphi]_i(z, \cdot)(t).$$

Proof. For the related homotopy formulae when H_i 's are replaced by the Leray-Koppelman homotopy operators, see Treves [50, VI.7.12, p. 294, VI.7.13, p. 294] for suitable q and [18] for the arbitrary q .

(i) Recall that the homotopy operator H_i are linear. To derive the homotopy formulae for \mathcal{D} , we will use the following estimates from (5.12) and (5.14):

$$|H_i[\varphi]_i|_{\overline{D};0} \leq C|[\varphi]_i|_{\overline{D};1}, \quad i = 0, 1, \dots, n.$$

Thus if $\psi_j \rightarrow \psi$ uniformly in $C^1(\overline{D})$ norm, then

$$(6.7) \quad \lim_{j \rightarrow \infty} H_i\psi_j = H_i\psi.$$

Also for $\psi \in C_*^{1,1}(\overline{D} \times \overline{S})$, we have

$$(6.8) \quad \frac{\partial}{\partial t_j} H_i\psi(\cdot, t) = H_i \frac{\partial}{\partial t_j} \psi(\cdot, t).$$

Analogously, if $\psi_j \rightarrow \psi$ in $C^0(\overline{S})$ as $j \rightarrow \infty$, then $\lim_{j \rightarrow \infty} R_i\psi_j = R_i\psi$. For $\psi \in C_*^{1,0}(\overline{D} \times \overline{S})$, we have $\frac{\partial}{\partial \bar{z}_j} R_i\psi = R_i \frac{\partial}{\partial \bar{z}_j} \psi$. Note that $\overline{\partial}$ commutes with the pull-back H^* of $H(t, \theta) = \theta t$. By (2.3), we have

$$(6.9) \quad d^0 H_i\varphi = -H_i d^0 \varphi, \quad \varphi \in C_*^{1,1}(\overline{D} \times \overline{S}),$$

$$(6.10) \quad \overline{\partial} R_i\varphi = -R_i \overline{\partial}\varphi, \quad \varphi \in C_*^{1,0}(\overline{D} \times \overline{S}).$$

Let us start with the integral representation of $[\varphi]_0$ in D . Since φ has total degree q , then $\deg_x[\varphi]_0 = q > 0$. We apply (5.10) for functions and the Poincaré formula for d^0 by (6.9). Thus for $[\varphi]_0 \in C_*^{1,1}$, we obtain in $D \times S$

$$(6.11) \quad [\varphi]_0 = H_0[\varphi]_0 + H_1 \overline{\partial}[\varphi]_0 = (d^0 R_q H_0[\varphi]_0 + R_{q+1} d^0 H_0[\varphi]_0) + H_1 \overline{\partial}[\varphi]_0.$$

Since $\overline{\partial}_z \Omega_{0,0}^1(z, \zeta) = 0$, we have $d^0 R_q H_0[\varphi]_0 = \mathcal{D} R_q H_0[\varphi]_0$. Combining with $d^0[\varphi]_0 = [\mathcal{D}\varphi]_0$, we express (6.11) as

$$(6.12) \quad [\varphi]_0 = \mathcal{D} R_q H_0[\varphi]_0 + R_{q+1} H_0[\mathcal{D}\varphi]_0 + H_1 \overline{\partial}[\varphi]_0.$$

Analogously, for $[\varphi]_j \in C_*^{1,1}(\overline{D} \times \overline{S})$, we get

$$(6.13) \quad [\varphi]_j = \overline{\partial} H_j[\varphi]_j + H_{j+1} \overline{\partial}[\varphi]_j = \mathcal{D} H_j[\varphi]_j - d^0 H_j[\varphi]_j + H_{j+1} \overline{\partial}[\varphi]_j.$$

By (6.9) and $d^0[\varphi]_j = [\mathcal{D}\varphi]_j - \bar{\partial}[\varphi]_{j-1}$, we obtain

$$\begin{aligned} \sum_{j>0} (-d^0 H_j[\varphi]_j + H_{j+1} \bar{\partial}[\varphi]_j) &= \sum_{j>0} (H_j [\mathcal{D}\varphi]_j - H_j \bar{\partial}[\varphi]_{j-1} + H_{j+1} \bar{\partial}[\varphi]_j) \\ &= -H_1 \bar{\partial}[\varphi]_0 + \sum_{j>0} H_j [\mathcal{D}\varphi]_j. \end{aligned}$$

Here we have used $H_{n+1} = 0$. Combining it with (6.12) and (6.13), we obtain

$$\varphi = \mathcal{D}R_q H_0[\varphi]_0 + R_{q+1} H_0[\mathcal{D}\varphi]_0 + \sum_{j>0} \mathcal{D}H_j[\varphi]_j + \sum_{j>0} H_j [\mathcal{D}\varphi]_j,$$

which gives us (i).

(ii) By $[\varphi]_j \in C^1(\bar{S} \times \bar{S})$ and the Poincaré lemma, we obtain

$$\varphi = [\varphi]_q + \sum_{i<q} (d^0 R_{q-i}[\varphi]_i + R_{q+1-i} d^0[\varphi]_i) = [\varphi]_q + \sum_{i<q} \mathcal{D}R_{q-i}[\varphi]_i + R_{q+1-i} \mathcal{D}[\varphi]_i.$$

Here we have used $\bar{\partial}R_{q-i}[\varphi]_i = -R_{q-i}\bar{\partial}[\varphi]_i$ for $i < q$ by (6.9) and $R_{q+1-i}d^0[\varphi]_i = R_{q+1-i}\mathcal{D}[\varphi]_i - R_{q-i}\bar{\partial}[\varphi]_i$. We express

$$\sum_{i<q} R_{q+1-i} \mathcal{D}[\varphi]_i = \sum_{i<q} R_{q+1-i} ([d^0\varphi]_i + [\bar{\partial}\varphi]_{i+1}) = -R_1[d^0\varphi]_q + \sum_{i\leq q} R_{q+1-i} [\mathcal{D}\varphi]_{i+1},$$

because $[\bar{\partial}\varphi]_0 = 0$. We have $[\varphi]_q(z, t) - R_1 d^0[\varphi]_q(z, \cdot)(t) = [\varphi]_q(z, 0)$. We now apply the homotopy formula (2.7) to express

$$[\varphi]_q(\cdot, 0) = \bar{\partial}H_q[\varphi]_q(\cdot, 0) + H_{q+1}\bar{\partial}[\varphi]_q(\cdot, 0) = \mathcal{D}H_q[\varphi]_q(\cdot, 0) + H_{q+1}([\mathcal{D}\varphi]_{q+1}(\cdot, 0)).$$

Combining the identities, we get (6.3) and (6.6). \square

Theorem 6.3. *Let $q > 0$. Let D be a strictly pseudoconvex domain with C^2 boundary. Let S be a bounded star-shaped domain in \mathbf{R}^m . Let φ be a \mathcal{D} -closed $(0, q)$ form in $C^1(\bar{D} \times \bar{S})$. Then there exists a solution $u \in C^1(\bar{D} \times \bar{S})$ to $\mathcal{D}u = \varphi$. Furthermore, the following properties hold.*

- (i) *Suppose that $[\varphi]_0 = 0$ additionally. If $\varphi \in \Lambda_*^{r,k}(\bar{D} \times \bar{S})$ with $k \in \{1, 2, \dots, \infty\}$ and $r \in (1, \infty]$, the v is in $\Lambda_*^{r+1/2,k}(\bar{D} \times \bar{S})$.*
- (ii) *Let $r \in [1, \infty]$. If $\varphi \in C^r(\bar{D} \times \bar{S})$, the u is in $C^r(\bar{D} \times \bar{S})$. If $\varphi \in \Lambda_r(\bar{D} \times \bar{S})$ and S is a Lipschitz domain, the u is in $\Lambda_r(\bar{D} \times \bar{S})$.*

Proof. (i) follows from (6.4) and (5.11). (ii) follows from (6.6), (6.1), and (5.11). \square

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